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AND

DISCUSSIONS

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AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

LINE LOAD ACTION ON THIN CYLINDRICAL SHELLS

BY HERMAN SCHORER,[1] ASSOC. M. AM. SOC. C. E.

SYNOPSIS

The object of this paper is the derivation of influence numbers for the longitudinal edge deformations of a thin, cylindrical, shell sector, rigidly held between transverse stiffening members. This problem is of practical importance in the design of modern shell structures, such as large pipe lines, self-supporting flume structures, sector gates, and the barrel type of roof described in this paper. The monolithic action of longitudinal stiffening members, as well as the analysis of discontinuous surface loads, leads to the consideration of corrective line loads, which restore the continuity of the structure.

The proposed method is based on simplifying assumptions, and, therefore, the solution is an approximation with definite limitations. As an illustrative example the secondary stress conditions in a half-full pipe line are investigated.

INTRODUCTION

Thin shells or bent plates are three-dimensional structures, extensively used by Nature as protective housings, on account of their great resistance to impact forces. They require a minimum of construction material and are adaptable to a great variety of possible shapes. The shells may be of the open or closed type.

The excellent structural properties of closed shells are explained by the fact that external surface loads are largely transmitted by direct stresses, which act in two principal stress planes. Although concentrated loads also induce bending stresses, Geckeler[2] has shown in the analysis of spherical shells that a direct stress system prevails within a short distance from

NOTE.—Discussion on this paper will be closed in August, 1935, *Proceedings*.

[1] Gen. Mgr., Bosan Tank Corp. of America, New York, N. Y.

[2] "Zur Theorie der Elastizität flacher rotations-symmetrischer Schalen," von J. W. Geckeler, *Ingenieur-Archiv*, Vol. 1, 1930, p. 255.

the load concentration. The bending zone appears distributed over an imaginary, circular plate, elastically supported by the adjacent parts of the shell. The free span of this plate is comparatively small and, therefore, the induced bending stresses are likewise small.

In case of open shells, the free rims or edges represent zones of structural weakness, because a direct stress system is possible in one direction only, namely, parallel to the free boundary. Edge loads, therefore, must be transmitted to the interior of the shell by partial plate action; that is, bending stresses in planes normal to the boundary line. For instance, a closed egg-shell represents a rather stiff structure. However, if the shell is cut into two halves, it will be observed that small edge forces cause relatively large distortions of the thin walls.

As a rule, Nature protects the free edges of thin shells by thickening the weak zone, as in the case of walnut shells. The same principle is used in the glass and sheet-metal industry, where thin edges are strengthened by means of flange-shaped stiffening members. In order to be most effective such members should be placed in planes nearly normal to the surface of the shell. The advantages of true shell action will then be realized also in the case of open shells.

The practical design of shell structures in steel and reinforced concrete has opened a wide field of interesting applications in industry and architecture. A description of the more recent developments in shell-type reinforced concrete structures is given by Dischinger and Finsterwalder.[3] Most of the practical problems lead to a consideration of edge loads in connection with the design of adequate stiffening members. For this reason the analysis of shell stresses due to edge loads becomes of practical importance. The following study is limited to the stresses and deformations in a circular, cylindrical, shell sector, with a uniform wall thickness and subject to edge loads distributed along a generatrix.

Closed cylindrical shells or tubes, subject to continuous surface loads, require circumferential stiffening members at the open ends, or over the supports, as in the case of large pipe lines,[4] horizontal storage tanks, culverts, etc. The actual stress condition in such a shell is determined by the superposition of two fundamental stress systems: First, the so-called membrane stresses (which represent a statically determinate, direct stress system, being in equilibrium with the surface loads); and, second, the so-called rim stresses (which are statically indeterminate) caused by the restraining action of the circumferential stiffeners. In the case of thin shells the rim stresses are distributed only over a comparatively narrow ring zone.

The analysis of the membrane stresses in symmetrical shells of revolution, due to continuous, symmetrical, and asymmetrical surface loads, has been well established and is presented in detail by Dischinger.[5]

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[3] "Neuere Entwicklungsformen der Schalenbauweise," von F. Dischinger und U. Finsterwalder, *Beton und Eisen*, Vol. 31 (1932).

[4] "Design of Large Pipe Lines," by Herman Schorer, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 101.

[5] "Handbuch für Eisenbeton," von H. J. Kraus und Franz Dischinger, Vol. 6, Fourth Edition, p. 269, Wilhelm Ernst & Sohn, Berlin, 1928.

A simplified method for the practical determination of rim stresses in symmetrical shells of revolution, subject to symmetrical rim loads, has been derived by Geckeler.[6] The rim load problem in circular, cylindrical shells, subject to asymmetrical loadings, has been fully treated by Miesel.[7]

Fig. 1 shows a cylindrical shell with a cross-section consisting of a thin, circular-ring sector, between two transverse end stiffeners, fixed in a plane normal to the axis of the shell. In the following derivations the term, "rim," always refers to the circumferential boundary of the shell and the term "edge," refers to the longitudinal boundary, which coincides with a generatrix of the cylindrical surface.

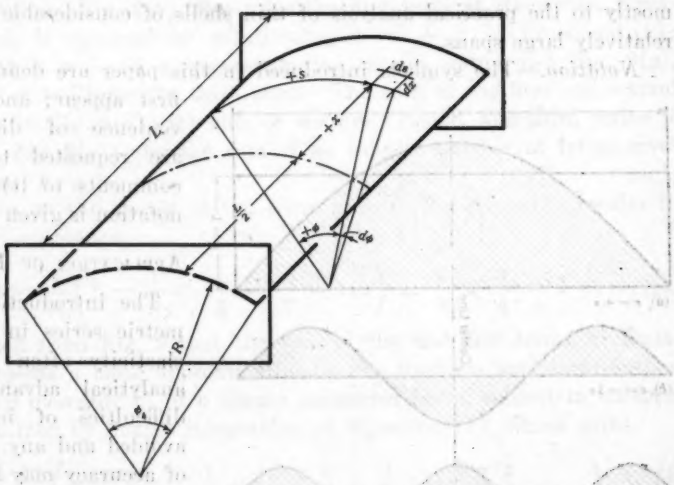

FIG. 1.

Line loads, or edge loads, are defined as loads acting on the two free edges of the shell. The present study is limited to continuous line loads only, such as those caused by the restraining action of a longitudinal edge-stiffening member. The shell stresses due to edge loads, then, represent a third stress system, designated as line load or edge stresses.

The edge load stresses in cylindrical shells, simply supported between stiff transverse members, have been investigated by Dr. Finsterwalder,[8] mainly for the purpose of analyzing a barrel, or vaulted, type of thin shell roof structure. Similar problems arise in the design of semi-circular, open flume structures, sector gates, decks of reinforced concrete dams, etc. The analysis of discontinuous surface loads, such as the partial fillings of a pipe line, partial submergence of submarines, lifting tanks, etc., also leads to the problem of line-load stresses.

[6] "Ueber die Festigkeit achsensymmetrischer Schalen," von J. Geckeler, *Forschungsarbeiten*, Heft 276, VDI-Verlag, Berlin, 1926.

[7] "Ueber die Festigkeit von Kreiszylinderschalen mit nichtachsensymmetrischer Belastung," von Kurt Miesel, *Ingenieur-Archiv*, Vol. 1, 1929, p. 22.

[8] "Die Theorie der zylindrischen Schalengewölbe, System Zeiss-Dywidag," von Dr. Ing. Ulrich Finsterwalder, International Assoc. for Bridge and Structural Eng., Zurich, 1932; also, "Die querversteiften zylindrischen Schalengewölbe mit kreissegmentförmigem Querschnitt," von Ulrich Finsterwalder, *Ingenieur-Archiv*, Vol. 4, 1933, p. 43.

In contrast to the comparatively narrow zone of circumferential rim load stresses, the influence of edge loads may extend over the entire shell area, because such loads are largely transmitted to the stiff transverse members. At present, no exact solution of the line-load problem is available. The approximate method presented in this paper is based entirely on Dr. Finsterwalder's original investigation, which also introduces simplifying assumptions, justified by model tests and observation of full-sized structures.

The basic assumptions point out definite limitations, which the designer must keep in mind when considering structures in which plate action, rather than shell action, comes into play. For this reason the method is applicable mostly to the practical analysis of thin shells of considerable curvature and relatively large spans.

Notation.—The symbols introduced in this paper are defined where they first appear; and for the convenience of discussers (who are requested to adapt their comments to it), the complete notation is given in Appendix I.

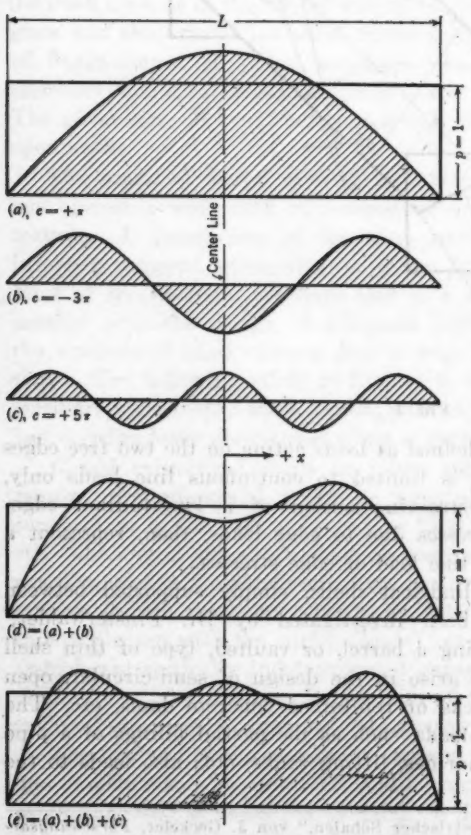

FIG. 2.—UNIFORM LOAD, $p = 1$, REPRESENTED BY FOURIER SERIES.

APPLICATION OF FOURIER SERIES

The introduction of trigonometric series in the theory of elasticity often yields decided analytical advantages because difficulties of integration are avoided and any desired degree of accuracy may be obtained by simply considering a sufficient number of terms. The solution of the flat slab problem by Lewe[9] and the derivation of the deflection curves of suspension bridges by Timoshenko,[10] are cited as well-known applications.

Consider a uniformly distributed unit load, p per unit length, acting on a simply supported beam of the span, L. This load distribution can be expressed by the sum of a trigonometric series. Assuming the origin of the co-ordinate system at the center of the span, then,

[9] "Pilzdecken," von Dr. Lewe, pub. by Wilhelm Ernst & Sohn, Berlin, 1926.

[10] "The Stiffness of Suspension Bridges," by S. Timoshenko, *Transactions*, Am. Soc. C. E., Vol. 94 (1930), p. 377.

$$p(x) = \frac{4}{\pi}\left(\cos\frac{\pi x}{L} - \frac{1}{3}\cos\frac{3\pi x}{L} + \frac{1}{5}\cos\frac{5\pi x}{L} - \dots\right) = (1) \dots (1)$$

in which, $p(x)$ = the sum of a trigonometric series expressing intensity of load; x = a horizontal distance measured longitudinally, from the center of the span; and L = the span of a simply supported beam or cylindrical shell. The general term of this series (Equation (1)) is,

$$p_c = \frac{4}{c}\cos\frac{c\,x}{L} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

in which, c = substitutions corresponding to the first, second, third, etc., term of a Fourier series. Fig. 2(a) illustrates the first term of the series, Equation (1), which is obtained by substituting $c = +\pi$ in Equation (2); Fig. 2(b), with $c = -3\pi$, represents the second term; and Fig. 2(c), with $c = +5\pi$, indicates the third term. The sum of the first and second terms is shown in Fig. 2(d); the sum of the first, second, and third terms is represented in Fig. 2(e); and the sum of an infinite number of terms gives the straight line, $p = 1$.

For example, at the center of the span, $x = 0$, Equation (1) results in the well-known expression,

$$p(0) = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) = \frac{4}{\pi} \times \frac{\pi}{4} = 1 \dots\dots (3)$$

It will be noted from Fig. 2 that the sum of the first few terms in Equation (1) represents a close approximation to the uniform load distribution.

The bending moment, M, of a simply supported beam, subject to uniform loading, is obtained by double integration of Equation (1), which gives,

$$M = \frac{4\,L^2}{\pi^3}\left(\cos\frac{\pi x}{L} - \frac{1}{3^3}\cos\frac{3\pi x}{L} + \frac{1}{5^3}\cos\frac{5\pi x}{L} - \dots\right) \dots\dots (4)$$

The general term of the series, Equation (4), is,

$$M_c = \frac{4}{c}\frac{L^2}{c^2}\cos\frac{c\,x}{L} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (5)$$

At the center of the span, $x = 0$, the sum of the series, Equation (5), gives the well-known moment value for a uniformly distributed unit loading, $M_0 = \frac{L^2}{8}$.

Likewise, the deflection, η, of a uniformly loaded and simply supported beam, is obtained by double integration of Equation (4); thus,

$$\eta = \frac{4\,L^4}{\pi^5}\left(\cos\frac{\pi x}{L} - \frac{1}{3^5}\cos\frac{3\pi x}{L} + \frac{1}{5^5}\cos\frac{5\pi x}{L} - \dots\right) \dots\dots (6)$$

in which, η = the deflection of a simply supported, uniformly loaded beam, multiplied by $E\,I$. The general term of the series, Equation (6), is,

$$\eta_c = \frac{4}{c}\frac{L^4}{c^4}\cos\frac{c\,x}{L} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (7)$$

At the center of the span, $x = 0$, the sum of the series, Equation (6), gives the value, $\eta_0 = \frac{5\ L^4}{384}$, which is identical with that derived by the customary methods.

Deformations of a Small Element

Fig. 3(*a*) shows a small element of a cylindrical shell, loaded by internal forces and couples as indicated, which are defined as stress components, measured per unit width of the element. The position of the element is defined by the distance, x, from the center of the span (see Fig. 1), and the angle, ϕ, measured from the left-hand edge of the shell when the element

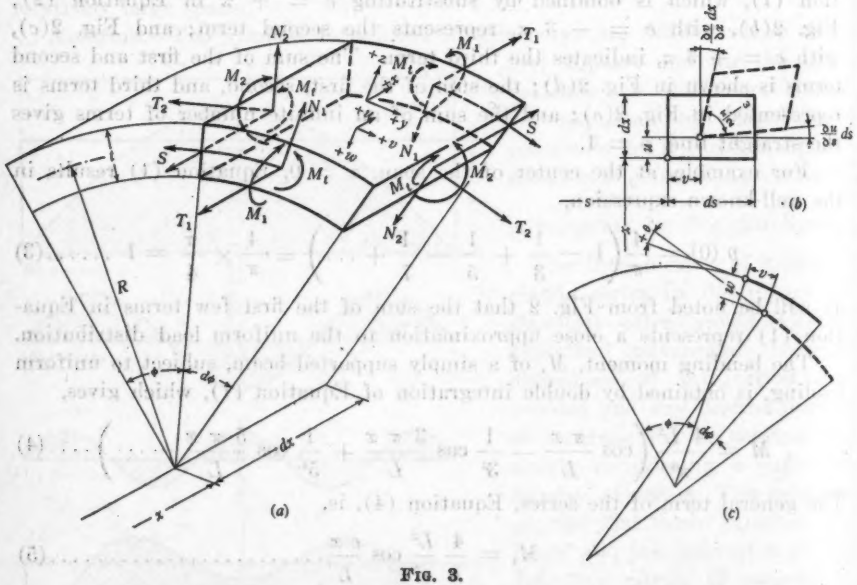

Fig. 3.

faces in the positive direction of x. The direct stress components, T_1, acting in a longitudinal direction, and, T_2, acting in a transverse direction, are positive when they produce tension. The bending stress components, M_1, acting in a longitudinal plane, and, M_2, acting in a transverse plane, are positive when they produce tension on the inner surface of the element. The direct shearing stress component, S, acting in pairs of equal magnitude on all four sides in the plane of the element, is considered positive when it produces tension in a diagonal direction of increasing values of x and ϕ. The radial shearing stress components, N_1 and N_2, are considered positive when they act in an outward direction on the sides facing the origin of x and ϕ, respectively. The torsional moments, or twisting moments, M_t, are practically equal on all four sides, and are defined as positive when they produce tension on the outside surface in a diagonal direction of increasing values of x and ϕ. For reasons stated subsequently the influence of the longitudinal

bending moment, M_1, the corresponding shearing stress component, N_1, and the influence of the torsional moments, M_t, will be neglected in the edge-load analysis.

The deflections of the small element are indicated in Figs. 3(*a*), 3(*b*), and 3(*c*). The longitudinal deflection, u, and the tangential deflection, v, are positive when the element moves in the positive direction of x and ϕ, respectively. The radial deflection, w, is positive when the element moves inward. The detrusion, ω, caused by the shearing stress component, S, is positive when S is positive. The angular deflection, θ, of the transverse tangent, is positive when the tangent is rotated in a clockwise direction.

Fig. 3 indicates the positive direction of all stress components and deflections. Love[11] has derived the following fundamental relations between the deformations, deflections, and stress components, applying to the small element of the length, dx; the width, $ds = R\,d\phi$; a constant thickness, t; and, a constant radius, R. Thus, the unit longitudinal elongation is expressed by:

$$e_1 = \frac{\partial u}{\partial x} = \frac{1}{E\,t}\,(T_1 - m\,T_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

the unit transverse elongation by,

$$e_2 = \frac{1}{R}\left(\frac{\partial v}{\partial \phi} - w\right) = \frac{1}{E\,t}\,(T_2 - m\,T_1) \quad \ldots\ldots\ldots\ldots (9)$$

the detrusion by,

$$\omega = \frac{\partial v}{\partial x} + \frac{\partial u}{R\,\partial \phi} = \frac{2\,S}{E\,t}\,(1 + m) \quad \ldots\ldots\ldots\ldots\ldots (10)$$

the angular deflection of the transverse tangent by,

$$E\,t\,\theta = \frac{1}{R}\left(\frac{\partial w}{\partial \phi} + v\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

the change of transverse curvature by,

$$\kappa_2 = \frac{1}{R^2}\left(\frac{\partial v}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2}\right) = \frac{1}{R^2}\left(w + \frac{\partial^2 w}{\partial \phi^2}\right) = -\frac{12\,(1 - m^2)\,M_2}{E\,t^3} \quad \ldots (12)$$

and the change of longitudinal curvature by,

$$\kappa_1 = \frac{\partial^2 w}{\partial x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

in which, e = unit elongation: e_1, longitudinal, and e_2, transverse; u, v, and w are, respectively, the longitudinal, tangential, and radial deflections of a shell element; E = modulus of elasticity; t = thickness of shell; T = direct stress components on a shell element: T_1, in a longitudinal direction, and T_2, in a tangential direction; m = Poisson's ratio; ω = angular detrusion of a shell element, caused by the shearing forces, S; R = radius of curvature of a cylinder element; ϕ = angular distance of a shell element from the left-

[11] "Theory of Elasticity," by A. E. H. Love, Third Edition, Cambridge Univ. Press (1920), p. 546.

hand edge; and κ = change of curvature of a shell element: κ_1, longitudinal, and κ_2, transverse.

The Edge Load Problem

Assume a thin cylindrical shell sector (such as that shown in Fig. 1)—of constant thickness, t; constant radius, R; a central angle, ϕ_k; and a span, L—between the two transverse end stiffeners. Assume, further, that no external loads are acting on the shell surface. Then consider a system of edge loads, such as thrusts, moments, shears, etc., continuously distributed along one or both of the free longitudinal boundaries, and resisted by the transverse stiffening members. Due to these edge loads the shell will be distorted by internal forces and couples (see Fig. 3), which transmit the loads to the end walls.

If the shell is thin, not extremely short, and has a fairly large central angle, experience and analysis both indicate that the internal work of the longitudinal bending moment, M_1, the corresponding shear, N_1, and the twisting moments, M_t, is rather small compared to the work done by the direct stress components, T_1, T_2, S, and the transverse bending moment, M_2, with the corresponding radial shearing stress component, N_2. Under these limiting conditions it will be permissible, for practical purposes, to neglect the influence of M_1, N_1, and M_t in considering the stresses and deformations of the shell due to edge loads. However, this statement does not apply to the ring zone adjacent to the end stiffeners, which is always subject to a supplementary stress system, defined by the rim stress zone, where M_1 and N_1 become relatively large, and M_2 and N_2 assume negligible values.

Based on the simplifying assumptions, $M_1 = N_1 = M_t = 0$, the five remaining stress components, M_2, N_2, T_2, T_1, and S, can then be derived from four equilibrium equations and one compatibility equation, as shown by Finsterwalder.[3] The edge load distribution is assumed to be symmetrical about the middle transverse plane, expressed by a cosine function, which can be considered as the general term of a trigonometric series. As suggested by Finsterwalder, the stress components are expressed conveniently by a stress function, $f(\phi)$, similar to Airy's presentation of the stress condition in a plane plate.

The stress function is introduced by the following definition of the bending stress component, M_2,

$$M_2 = -\cos\frac{c\,x}{L}\,f(\phi) \qquad (14)$$

Fig. 3(a) gives the following equilibrium conditions on a small element:

Rotation in the transverse plane,

$$N_2 = \frac{\partial M_2}{R\,\partial\phi} = -\frac{1}{R}\,\frac{df}{d\phi}\cos\frac{c\,x}{L} \qquad (15)$$

projection on the z-axis,

$$T_2 = -\frac{\partial N_2}{\partial\phi} = +\frac{1}{R}\,\frac{d^2f}{d\phi^2}\cos\frac{c\,x}{L} \qquad (16)$$

projection on the y-axis, and by partial integration,

$$\frac{\partial S}{\partial x} = \frac{N_2}{R} - \frac{1}{R}\frac{\partial T_2}{\partial \phi} = -\frac{1}{R^2}\left(\frac{df}{d\phi} + \frac{d^3f}{d\phi^3}\right)\cos\frac{c\,x}{L} \quad \ldots\ (17)$$

or,

$$S = -\frac{L}{R^2\,c}\left(\frac{df}{d\phi} + \frac{d^3f}{d\phi^3}\right)\sin\frac{c\,x}{L} \quad \ldots\ (18)$$

and projection on the x-axis, and by partial integration,

$$\frac{\partial T_1}{\partial x} = -\frac{1}{R}\frac{\partial S}{\partial \phi} = \frac{L}{R^2\,c}\left(\frac{d^2f}{d\phi^2} + \frac{d^4f}{d\phi^4}\right)\sin\frac{c\,x}{L} \quad \ldots\ (19)$$

or,

$$T_1 = -\frac{L^2}{R^2\,c^2}\left(\frac{d^2f}{d\phi^2} + \frac{d^4f}{d\phi^4}\right)\cos\frac{c\,x}{L} \quad \ldots\ (20)$$

The following definition is introduced,

$$a = \frac{c^2\,R^2}{L^2} \quad \ldots\ (21)$$

Substituting T_2 and T_1 from Equations (16) and (20), in Equation (8), the longitudinal deflection, u, is obtained by partial integration,

$$E\,t\,u = -\frac{1}{a\sqrt{a}}\left(\frac{d^2f}{d\phi^2}(1 + m\,a) + \frac{d^4f}{d\phi^4}\right)\sin\frac{c\,x}{L} \quad \ldots\ (22)$$

From Equation (10), by partial differentiation,

$$\frac{\partial^2 v}{\partial x^2} = \frac{2\,(1+m)}{E\,t}\frac{\partial S}{\partial x} - \frac{\partial^2 u}{\partial s\,\partial x} \quad \ldots\ (23)$$

By partial differentiation of Equation (8), with respect to s,

$$\frac{\partial^2 u}{\partial s\,\partial x} = \frac{1}{E\,t}\left(\frac{\partial T_1}{\partial s} - m\frac{\partial T_2}{\partial s}\right) \quad \ldots\ (24)$$

Substituting Equation (24) in Equation (23),

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{E\,t}\left(2\,(1+m)\frac{\partial S}{\partial x} - \frac{\partial T_1}{R\,\partial\,\phi} + m\frac{\partial T_2}{R\,\partial\,\phi}\right) \quad \ldots\ (25)$$

From Equations (16), (17), and (20), the expressions for the stress components are substituted in Equation (25), then, by partial integration,

$$E\,t\,v = \frac{1}{a}\left[2(1+m)\left(\frac{df}{d\phi} + \frac{d^3f}{d\phi^3}\right) - \frac{1}{a}\left(\frac{d^3f}{d\phi^3} + \frac{d^5f}{d\phi^5}\right) - m\frac{d^3f}{d\phi^3}\right]\cos\frac{c\,x}{L} \quad \ldots\ (26)$$

From Equation (9), with T_2 from Equation (16), T_1 from Equation (20), and $\frac{\partial v}{\partial \phi}$ from Equation (26),

$$E\,t\,w = \left[\frac{d^2f}{d\phi^2}\left(\frac{m}{a}-1\right)+\frac{2}{a}\left(\frac{d^2f}{d\phi^2}+\frac{d^4f}{d\phi^4}\right)\right.$$
$$\left.-\frac{1}{a^2}\left(\frac{d^4f}{d\phi^4}+\frac{d^6f}{d\phi^6}\right)\right]\cos\frac{c\,x}{L} \quad \ldots\ldots\ldots\ldots\ldots (27)$$

From Equation (11), with reference to Equations (26) and (27),

$$E\,t\,R\,\theta = \left[-\frac{d^3f}{d\phi^3}+\frac{2}{a}\left(\frac{df}{d\phi}+2\frac{d^3f}{d\phi^3}+\frac{d^5f}{d\phi^5}\right)\right.$$
$$\left.-\frac{1}{a^2}\left(\frac{d^3f}{d\phi^3}+2\frac{d^5f}{d\phi^5}+\frac{d^7f}{d\phi^7}\right)+2\frac{m}{a}\left(\frac{df}{d\phi}+\frac{d^3f}{d\phi^3}\right)\right]\cos\frac{c\,x}{L} \quad \ldots. (28)$$

The constants of integration are zero, provided the following conditions are fulfilled: First, the load distribution is symmetrical about the middle transverse plane, $x = 0$; second, the shell can deform freely at the end stiffeners in a longitudial and also in a radial direction. Since the present analysis is limited to edge loads expressed by a cosine function the first condition is met. The transverse stiffening members, as a rule, are relatively flexible in a longitudinal plane, so that free movement in this direction can be assumed for practical purposes. However, the movement in a radial direction is greatly resisted by the stiffness of the transverse members, requiring the consideration of rim stresses, which can always be treated as a separate stress system. For the present purpose, therefore, the loads can be assumed as transmitted to the end members by means of tangential shearing stresses only, in accordance with the membrane theory.

The compatibility equation is obtained by substituting M_2 from Equation (14), and w from Equation (27), in the expression for the transverse curvature, Equation (12), which gives the differential equation of the stress function,

$$\frac{d^8f}{d\phi^8}+2(1-a)\frac{d^6f}{d\phi^6}+[1-(4+m)\,a+a^2]\frac{d^4f}{d\phi^4}$$
$$+[-(2+m)\,a+a^2]\frac{d^2f}{d\phi^2}+4\,a^2b^2(1-m^2)f=0 \quad \ldots\ldots. (29)$$

The constant, b, is defined as follows,

$$b=\sqrt{3}\,\frac{R}{t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

The general solution of Equation (29) gives the following expression for the stress function,

$$f(\phi) = e^{(J1)\phi}(A\sin K_1\phi + B\cos K_1\phi)$$
$$+ e^{(J1)'\phi}(C\sin K'_1\phi + D\cos K'_1\phi) \ldots\ldots\ldots\ldots\ldots (31)$$

Equation (31) indicates that the stress function consists of the sum of two damped waves, in which, $(J1)$ and $(J1)'$, represent the damping factors, and K_1 and K'_1 determine the wave length. These four values are constants,

entirely governed by the dimensions of the shell (as exponents, $(J1)$ and $(J1)'$ correspond with J_1 and J'_1 in the text). The expressions, A, B, C, and D, are also constants, determined by the boundary conditions of the longitudinal edge, by which four stress components or deflections are assigned specified values, given by the edge load conditions or edge deformations.

The shell constants, J and K, are found by substituting the derivatives of Equation (31) in the differential formula, Equation (29). The first term of the stress function leads to the following relation:

$$e^{(J1)\phi} \sin K_1 \phi \left\{ J_8 + 2(1-a) J_6 + [1 - (4+m)a + a^2] J_4 + a(-2-m+a) J_2 + 4a^2 b^2 (1-m^2) \right\} + e^{(J1)\phi} \cos K_1 \phi \left\{ K_8 + 2(1-a) K_6 + [1-(4+m)a+a^2] K_4 + a(-2-m+a) K_2 \right\} = 0 \quad \ldots\ldots(32)$$

The expressions, J_2, J_4, etc., and K_2, K_4, etc., are defined by the general progressive functions,

$$J_n = J_1 J_{n-1} - K_1 K_{n-1} \quad \ldots\ldots(33)$$

and,

$$K_n = J_1 K_{n-1} + K_1 J_{n-1} \quad \ldots\ldots(34)$$

Equation (32) can be satisfied when the functions, F and G, defined by Equations (35) and (36), are zero,

$$F = J_8 + 2(1-a) J_6 + [1-(4+m)a+a^2] J_4 + a(-2-m+a) J_2 + 4a^2 b^2 (1-m^2) = 0 \quad \ldots\ldots(35)$$

and,

$$G = K_8 + 2(1-a) K_6 + [1-(4+m)a+a^2] K_4 + a(-2-m+a) K_2 = 0 \quad \ldots\ldots(36)$$

The derivatives of the second term of Equation (31), substituted in Equation (29), give an identical pair of equations for F and G.

The progressive functions, Equations (25) and (26), furnish the following expressions,

$$J_2 = J^2_1 - K^2_1 \quad \ldots\ldots(37)$$

$$J_4 = J^2_2 - K^2_2 \quad \ldots\ldots(38)$$

$$J_6 = J^2_2 - 3 J_2 K^2_2 \quad \ldots\ldots(39)$$

$$J_8 = J^4_2 - 6 J^2_2 K^2_2 + K^4_2 \quad \ldots\ldots(40)$$

$$K_2 = 2 J_1 K_1 \quad \ldots\ldots(41)$$

$$K_4 = 2 J_2 K_2 \quad \ldots\ldots(42)$$

$$K_6 = 3\,J^2_2\,K_2 - K^3_2 \ldots\ldots\ldots\

With these simplifying assumptions the differential equation (Equation (29)) is now reduced to the following expression,

$$\frac{d^8 f}{d\phi^8} + 4\,a^2\,b^2\,(1 - m^2)\,f = 0 \qquad (47)$$

The corresponding simultaneous equations, (Equations (35) and (36)), with J_8 and K_8 from Equations (40) and (44) respectively, become, therefore,

$$J^4_2 - 6\,J^2_2\,K^2_2 + K^4_2 + 4\,a^2\,b^2\,(1 - m^2) = 0 \qquad (48)$$

and,

$$4\,J^3_2 - 4\,J_2\,K^2_2 = 0 \qquad (49)$$

which gives,

$$J^2_2 = K^2_2 = a\,b\,\sqrt{1 - m^2} \qquad (50)$$

The following definitions are introduced:

$$r = \sqrt{a\,b}\ \sqrt[4]{1 - m^2} = \frac{c\,R}{L}\sqrt{\frac{R}{d}}\ \sqrt[4]{3\,(1 - m^2)} \qquad (51)$$

$$n_1 = \left(\sqrt{0.5} + 0.5\right)^{\frac{1}{2}} \qquad (52)$$

and,

$$n_2 = \left(\sqrt{0.5} - 0.5\right)^{\frac{1}{2}} \qquad (53)$$

The solution of Equation (50) gives the following roots for the unknown, J_2:

$$J_2 = + r \qquad (54)$$

and,

$$J'_2 = - r \qquad (55)$$

The corresponding values for the unknown, K_2, are,

$$K_2 = - r \qquad (56)$$

and,

$$K'_2 = - r \qquad (57)$$

The values, J_1, J'_1, K_1, K'_1, as well as J_3, J'_3, K_3, K'_3, etc., are found by means of the progressive functions, Equations (33) and (34). Since only waves of decreasing amplitude can describe the stress conditions originating from a loaded edge, it is necessary to choose the negative sign for the damping factors, J_1 and J'_1. This requirement, in turn, determines the choice of the negative sign for K_2 and K'_2, as given by Equations (56) and (57). Table 1 (Appendix II), shows the values, J, K, J', and K', respectively, corresponding to the simplified differential formula (Equation (47)).

By Newton's method it is now possible to find corrections for J_2 and K_2, designated by ΔJ_2 and ΔK_2. For this purpose the approximate values of

J and K, as given by Table 1 (Appendix II) are substituted in Equations (35) and (36), whereby the errors, Δ_1 and Δ_2, are obtained:

$$\Delta_1 = -4r^3(1-a) - ar(2+m-a) \quad \ldots\ldots(58)$$

and,

$$\Delta_2 = +4r^2(1-a) - a(2+m-a) + 2r(1-(4+m)a+a^2) \quad \ldots\ldots(59)$$

By partial differentiation of Equations (35) and (36), with respect to J_2 and K_2, the functions, f_1, f_2, and g_1, g_2, result,

$$\frac{\partial F}{\partial J_2} = f_1(J_2, K_2) \quad \ldots\ldots(60)$$

$$\frac{\partial F}{\partial K_2} = f_2(J_2, K_2) \quad \ldots\ldots(61)$$

$$\frac{\partial G}{\partial J_2} = g_1(J_2, K_2) \quad \ldots\ldots(62)$$

and,

$$\frac{\partial G}{\partial K_2} = g_2(J_2, K_2) \quad \ldots\ldots(63)$$

The approximate values for J_2 and K_2, as given by Equations (54) and (56), are now substituted in Equations (60) to (63), whereby the following equations of differences are obtained:

$$\Delta F_1 = \Delta J_2 \times f_1(r) \quad \ldots\ldots(64)$$

$$\Delta F_2 = \Delta K_2 \times f_2(r) \quad \ldots\ldots(65)$$

$$\Delta G_1 = \Delta J_2 \times g_1(r) \quad \ldots\ldots(66)$$

and,

$$\Delta G_2 = \Delta K_2 \times g_2(r) \quad \ldots\ldots(67)$$

in which, r is a shell constant and $f(r)$ and $g(r)$ are functions of r. The desired corrections, ΔJ_2 and ΔK_2, can then be determined from the two simultaneous equations:

$$\Delta F_1 + \Delta F_2 = \Delta J_2 \times f_1(r) + \Delta K_2 \times f_2(r) = -\Delta_1 \quad \ldots\ldots(68)$$

and,

$$\Delta G_1 + \Delta G_2 = \Delta J_2 \times g_1(r) + \Delta K_2 \times g_2(r) = -\Delta_2 \quad \ldots\ldots(69)$$

The same method is applied in order to find corrections for the approximate values, J'_2 and K'_2, in Equations (55) and (57). Neglecting the terms of a lower order (which is justified when $a < r$) the corrected values are:

$$J_2 = +r + \Delta J_2 = +r + \frac{(a-1)}{2} \quad \ldots\ldots(70)$$

$$K_2 = -r + \Delta K_2 = -r \quad \ldots\ldots(71)$$

$$J'_2 = -r + \Delta J'_2 = -r + \frac{(a-1)}{2} \quad \ldots\ldots(72)$$

and,

$$K'_2 = -r + \Delta K'_2 = -r \quad \ldots\ldots(73)$$

If necessary, additional corrections could be obtained by repeated applications of Newton's method. Since the general expressions for the second correction are rather complicated it is preferable to substitute the numerical values in individual cases. The final values then are substituted in Equations (37) to (44), in such a way that the entire group can be determined with any desired degree of accuracy. However, the more accurate solution of the problem, as carried through by Finsterwalder, cannot be stated in explicit form. In addition, the numerical computations must be repeated for each individual load term, c, so that, for a complete solution, the work is considerable.

It seems desirable, therefore, to obtain approximate solutions in simplified form, which are entirely sufficient in many cases, especially for the purpose of preliminary designs and estimates. If the shell is comparatively thin $\left(\frac{R}{t} > 100\right)$ and the relative span remains within the approximate limits, $1 < \frac{L}{R} < 10$, it can be shown that the correction, $\frac{(a-1)}{2}$ in Equations (70) and (72), does not exceed about $0.1r$. Under these limiting conditions it is then permissible to use the first approximation of the values, J and K, given in Table 1 (Appendix II), as a basis for an approximate method.

Consistent with the reasoning stated in arriving at the simplified differential equation (Equation (47)), the derivations of a lower order may also be neglected in the expressions for the stress components and deformations, Equations (14) to (28). With the limiting assumption, $a < 1$, it will be noted that the derivative of the highest order in each case also has the largest coefficient.

In order to obtain identical conditions in a longitudinal direction, only the stress components and deflections containing the factor, $\cos \frac{c\,x}{L}$, will be considered in the simultaneous elastic equations. For this reason the stress component, $\frac{\partial S}{\partial x}$ (Equation (17)), is introduced instead of S (Equation (18)); likewise, the stress component, T_1 (Equation (20)), is introduced instead of the deflection, u (Equation (22)). Assuming that continuity of stresses and deformations has been established on any point of a generatrix, the same condition then applies throughout.

The equations for the stress components and deflections are written in the general form,

$$(A\ a^* + B\ b^* + C\ c^* + D\ d^*)\ \cos\frac{c\,x}{L} \quad \ldots\ldots\ldots\ldots (74)$$

in which, with reference to Equation (31), the constants, A, B, C, and D, must conform to four edge conditions, at $\phi = 0$, for example, four known values of the stress components, M_2, N_2, T_2, and $\frac{\partial S}{\partial x}$, or, the three deflections, v, w, θ, and the stress component, T_1.

The expressions, a^*, b^*, c^*, and d^*, are functions of the angle, ϕ, generally defined as follows,

$$a^* = e^{(J1)\phi}\ (\alpha \sin K_1\,\phi + \beta \cos K_1\,\phi) \quad \ldots\ldots\ldots\ldots (75)$$

$$b^* = e^{(J1)\phi}\ (\alpha \cos K_1\,\phi - \beta \sin K_1\,\phi) \quad \ldots\ldots\ldots\ldots (76)$$

$$c^* = e^{(J1)'\phi}\ (\gamma \sin K'_1\,\phi + \delta \cos K'_1\,\phi) \quad \ldots\ldots\ldots\ldots (77)$$

and,

$$d^* = e^{(J1)'\phi}\ (\gamma \cos K'_1\,\phi - \delta \sin K'_1\,\phi) \quad \ldots\ldots\ldots\ldots (78)$$

Based on the simplifying assumptions regarding the derivatives of lower order, in case, $a < 1$, the constants, α, β, γ, and δ, are derived from Equations (14), (15), (16), (17), (20), (26), (27), and (28), in connection with Table 1 (Appendix II). The value of these constants, corresponding to the various stress components and deflections, is shown in Table 2 (Appendix II), where,

$$k = \sqrt{2} - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (79)$$

Influence Numbers for Edge Deformations

The practical analysis of edge load stresses is greatly facilitated by the use of influence numbers for the edge deformations, corresponding to elementary, unit loads, herein defined by the general term of the trigonometric series, Equation (2). All deformations must be given for the center of the span, $x = 0$, where the intensity of the unit load assumes the value, $\frac{4}{c}$, measured per unit length.

The four fundamental edge loadings to be considered herein, consisting of $M_2 = \frac{4}{c}$; $N_2 = \frac{4}{c}$; $T_2 = \frac{4}{c}$; and $\frac{\partial S}{\partial x} = \frac{4}{c}$, are preferably divided in symmetrical and contra-symmetrical load cases, as shown in Fig. 4. ("Contra-symmetrical" denotes "equal but in opposite directions.") In both cases the same absolute load is acting at the edges, $\phi = 0$ and $\phi = \phi_k$, with the positive direction at $\phi = 0$. The contra-symmetrical loading

consists in reversing the directions of the loads at $\phi = \phi_k$. In all cases the deformations are given for the left-hand edge, $\phi = 0$.

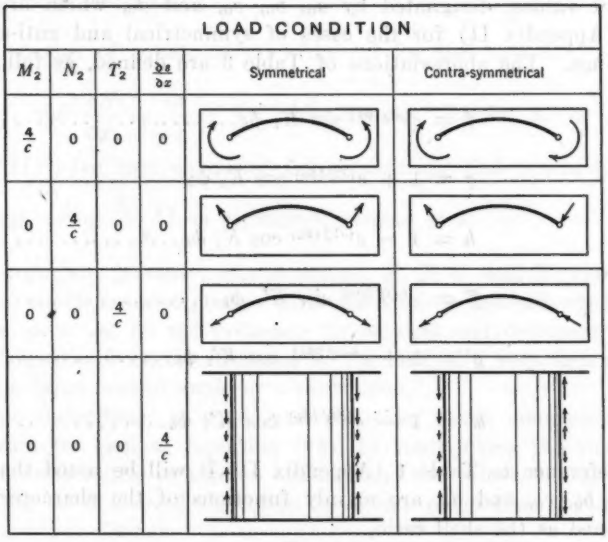

FIG. 4.

The stress components and deformations at any point of the shell sector, located by the angle, ϕ_1 (Fig. 5), are generally influenced by the stress waves from both the loaded edges. In Equations (75) to (78) the influence of the loads at $\phi = 0$ is found by means of the substitution, $\phi = \phi_1$. Likewise, the substitution, $\phi = \phi_2$, gives the influence of the loads acting at the edge, $\phi = \phi_k$. The algebraic sum of the two values represents the combined influence for the point, ϕ_1, and is designated by a^*_1, b^*_1, c^*_1, and d^*_1. Therefore,

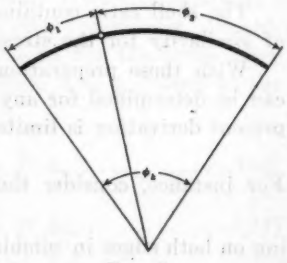

FIG. 5.

$$a^*_1 = a^*(\phi_1) \pm a^*(\phi_2) \quad \ldots\ldots(80)$$

$$b^*_1 = b^*(\phi_1) \pm b^*(\phi_2) \quad \ldots\ldots(81)$$

$$c^*_1 = c^*(\phi_1) \pm c^*(\phi_2) \quad \ldots\ldots(82)$$

and,

$$d^*_1 = d^*(\phi_1) \pm d^*(\phi_2) \quad \ldots\ldots(83)$$

With reference to Fig. 3(a), showing the positive directions of the stress components and deflections, it should be noted that, in case of symmetrical edge load conditions, the upper sign in Equations (80) to (83) applies to M_2, T_2, T_1, and w, and the lower sign to N_2, $\frac{\partial S}{\partial x}$, v, and θ. In case of anti-symmetrical loadings the corresponding signs must be reversed.

Considering the stress components and deflections on the left-hand edge, $\phi = 0$, the substitutions, $\phi_1 = 0$, $\phi_2 = \phi_k$, in Equations (80) to (83), furnish a group of values, designated by a_0, b_0, c_0, and d_0, which are shown in Table 3 (Appendix II) for the cases of symmetrical and anti-symmetrical edge loadings. The abbreviations of Table 3 are defined, as follows,

$$f = e^{(J1)(\phi k)} \sin K_1 \phi_k \quad \ldots\ldots(84)$$

$$g = 1 + e^{(J1)(\phi k)} \cos K_1 \phi_k \quad \ldots\ldots(85)$$

$$h = 1 - e^{(J1)(\phi k)} \cos K_1 \phi_k \quad \ldots\ldots(86)$$

$$f' = e^{(J1)'(\phi k)} \sin K'_1 \phi_k \quad \ldots\ldots(87)$$

$$g' = 1 + e^{(J1)'(\phi k)} \cos K'_1 \phi_k \quad \ldots\ldots(88)$$

and,

$$h' = 1 - e^{(J1)'(\phi k)} \cos K'_1 \phi_k \quad \ldots\ldots(89)$$

With reference to Table 1 (Appendix II) it will be noted that the edge values, a_0, b_0, c_0, and d_0, are mainly functions of the characteristic number, λ, defined as the shell ratio,

$$\lambda = \sqrt{r}\,\phi_k \quad \ldots\ldots(90)$$

The shell ratio contains all dimensions of the shell and expresses the law of similarity for the stress distribution in the case of edge loads.

With these preparations the values, A, B, C, and D in Equation (74), can be determined for any desired combination of four edge conditions. The present derivation is limited to the fundamental loadings, as shown in Fig. 4.

For instance, consider the case of a symmetrical unit load, $M_2 = \frac{4}{c}$, acting on both edges in combination with the specification, $N_2 = T_2 = \frac{\partial S}{\partial x} = 0$.

From Table 3(a) (Appendix II) (by using the subscripts, 1, 2, 3, etc., in reference to the edge values of M_2, N_2, T_2, etc.), the following equations can be written:

$$M_2 = A\,a_{01} + B\,b_{01} + C\,c_{01} + D\,d_{01} = \frac{4}{c} \quad \ldots\ldots(91)$$

$$N_2 = A\,a_{02} + B\,b_{02} + C\,c_{02} + D\,d_{02} = 0 \quad \ldots\ldots(92)$$

$$T_2 = A\,a_{03} + B\,b_{03} + C\,c_{03} + D\,d_{03} = 0 \quad \ldots\ldots(93)$$

and,

$$\frac{\partial S}{\partial x} = A\,a_{04} + B\,b_{04} + C\,c_{04} + D\,d_{04} = 0 \quad \ldots\ldots(94)$$

The solution of these four simultaneous equations furnishes the unknowns, A, B, C, and D, which can be stated in explicit form for any given numerical value of the shell ratio, λ.

Similar groups of equations are obtained for the unit loads, $N_2 = \frac{4}{c}$, $T_2 = \frac{4}{c}$, and $\frac{\partial S}{\partial x} = \frac{4}{c}$. In this manner Tables 4, 5, 6, and 7 (Appendix II) have been computed for symmetrical and contra-symmetrical unit loadings, using $\frac{K_1 \phi_k}{\pi}$ as argument, instead of λ.

As the argument increases, the functions, A, B, C, and D, approach constant values, which are also given. This condition is reached when the loads at the edge, $\phi = \phi_k$, do not influence the stresses and deformations of the opposite edge, $\phi = 0$ as, for instance, in the case of a very thin shell, combined with a large central angle or a short span.

The edge deflections, v, w, θ, and the edge stress component, T_1, are obtained from the general Equation (74), by substituting the values, A, B, C, and D, corresponding to a given loading, and the values, a_0, b_0, c_0, and d_0, corresponding to a desired deflection or stress component. For instance, consider again the case of a symmetrical unit loading, $M_2 = \frac{4}{c}$, for which the values, A, B, C, and D, are given by Table 4(a) (Appendix II). From Table 3(a) (Appendix II) with the subscripts previously defined, Equation (74) then furnishes the following expressions for the stress component, T_1, and the deflections at the center of the span, $x = 0$, defined as the influence numbers for M_2,

$$T_1 = A\, a_{08} + B\, b_{08} + C\, c_{08} + D\, d_{08} \quad \ldots\ldots\ldots\ldots\ldots (95)$$

$$v = A\, a_{08} + B\, b_{08} + C\, c_{08} + D\, d_{08} \quad \ldots\ldots\ldots\ldots\ldots (96)$$

$$w = A\, a_{07} + B\, b_{07} + C\, c_{07} + D\, d_{07} \quad \ldots\ldots\ldots\ldots\ldots (97)$$

and,

$$\theta = A\, a_{08} + B\, b_{08} + C\, c_{08} + D\, d_{08} \quad \ldots\ldots\ldots\ldots\ldots (98)$$

Similar expressions are obtained for the other unit loadings. The influence numbers, as shown in Table 8 (Appendix II), can be expressed in explicit form, except for a numerical coefficient, which is a function of the shell ratio only.

It is interesting to note that the influence numbers for the deflections, θ, w, and v, are symmetrical about the diagonal through the top left corner. The same symmetry applies to the coefficients, c_{21}, c_{12}, c_{31}, c_{13}, etc., throughout the diagonal, because the computations show that $c_{12} = c_{21}$, $c_{13} = c_{31}$, etc., in accordance with Maxwell's theorem of reciprocal relations between loads

and deflections. Since T_1 represents a stress component, instead of a deflection, the symmetrical relations apply to the coefficients only, not to the influence numbers for T_1 as a whole.

Table 9 (Appendix II) gives the values of the influence number coefficients for symmetrical and contra-symmetrical loading, with $\frac{K_1 \phi_k}{\pi}$ as the argument. These coefficients, plotted on logarithmic paper, are also shown in Fig. 6. It will again be noted that, for large values of the shell ratio, the coefficients approach constant values, indicating that the deflections of the edge, $\phi = 0$, become independent from the loads acting at the opposite edge, $\phi = \phi_k$. For medium values of the shell ratio the coefficients have a pronounced damped-wave shape, caused by the interference of stress waves originating from the loaded edges, which indicates typical shell action.

For smaller values of the shell ratio the coefficients plot on straight lines, indicating that plate action, rather than shell action, prevails in this region. In order to illustrate this point, consider a comparatively long shell having a small central angle, ϕ_k, and comparatively thick walls. The behavior of such a shell approaches that of a narrow, flat plate, spanning between the end stiffeners. Assume that the shell is loaded by symmetrical edge moments, M_2, which loading represents an equilibrium system in a transverse plane. Since the shell is of such a shape that the restraining action of the transverse end stiffeners becomes negligible, it is evident that the cross-section of the shell can be considered as a slightly curved beam, loaded by two equal end moments. The angular deflection, θ, can then be computed by the usual beam theory. A comparison with the values obtained from the influence number of the shell shows close agreement in this case. The agreement is also fairly good for the deflections, w and v. The longitudinal stress component, T_1, becomes negligible, which is also in accordance with plate action. Of course, the reason for the satisfactory agreement under this extreme condition is due to the fact that the transverse moment, M_2, is included in the stress components acting on the small element, Fig. 3(a).

On the other hand, consider the same shell loaded by a system of antisymmetrical edge moments, M_2. The behavior of the narrow shell strip then approaches that of a plate or a rod, held at both ends, and subjected to torsional moments distributed over the entire length. Evidently, the torsional deflections will govern entirely in this case. However, since the torsional moments, acting on a small shell element, have been neglected in the derivation of the differential equation (Equation (29)), it cannot be expected that the influence numbers (Table 8 (Appendix II)) would give even a rough approximation in this case of extreme conditions. The actual deflections will be much smaller than those given by the influence numbers, because the torsional work comes into play.

Similar considerations apply to other unit loadings when the central angle becomes rather small. For instance, a symmetrical load system, N_2, would then induce comparatively large bending moments, M_1, in a longitudinal direction, which have also been neglected. For these reasons the influence

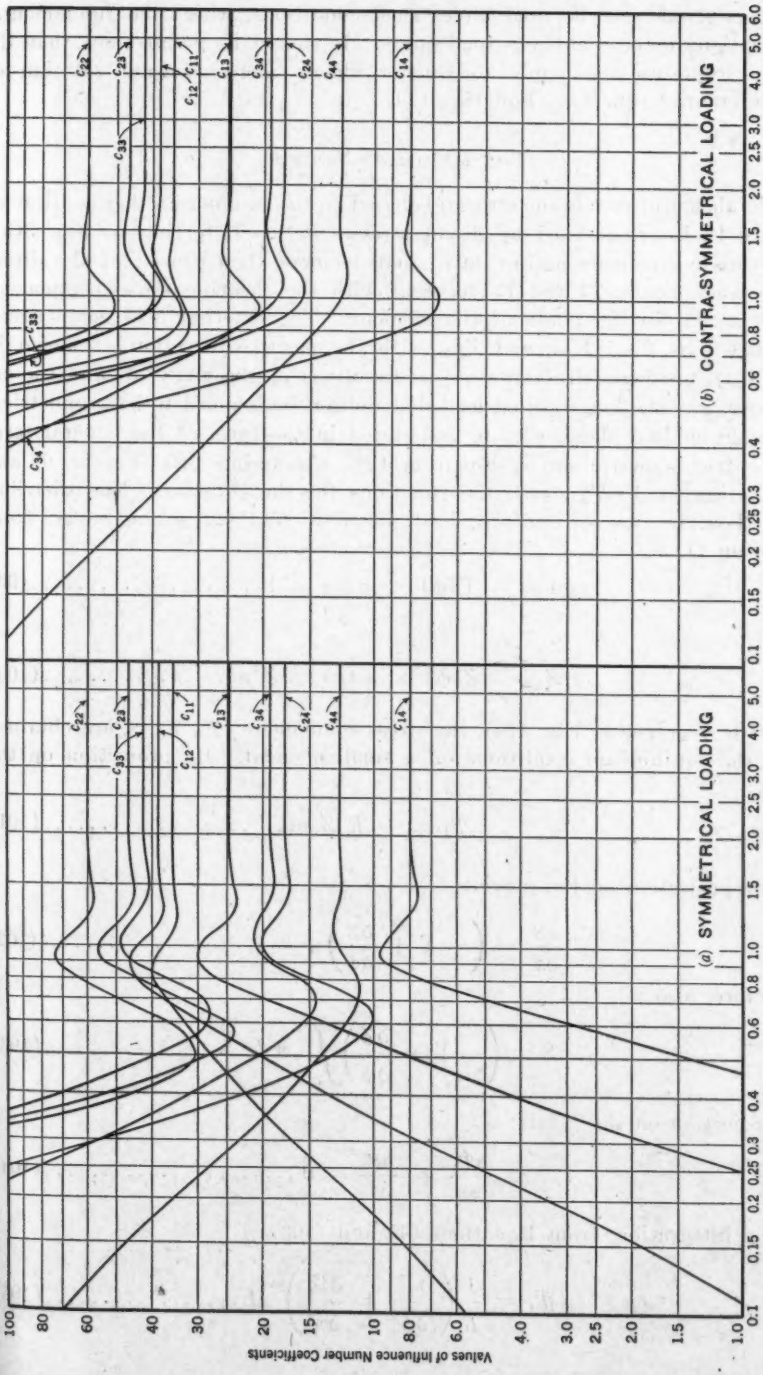

FIG. 6.—INFLUENCE NUMBER COEFFICIENTS.

numbers should not be used under such conditions, where the fundamental assumptions become entirely inadequate. It should be pointed out that the same exceptions also apply to Finsterwalder's more accurate solution of the differential equation (Equation (29)).

The Membrane System

The theory of membrane stresses is based on the assumption that continuous surface loads are resisted by direct stresses only. Referring to Fig. 3(*a*), the stress components acting on a small element then consist of the direct stress components, T_1 and T_2, together with the shearing stress component, S, all acting in the plane of the element. The external load components, designated by $X_{x\phi}$, $Y_{x\phi}$, and $Z_{x\phi}$, with the positive direction as shown in Fig. 3(*a*), are generally expressed as functions of the angle, ϕ, and the coordinate, x. The longitudinal load distribution is assumed to be symmetrical about the middle plane, $x = 0$, and stated in the form of the general term of the trigonometric series, Equation (2). Assuming that $X_{x\phi} = 0$, and designating by $Y(\phi)$, $Z(\phi)$, the functions for the transverse load distribution, then, in case of uniform load distribution along a generatrix, from Equation (1),

$$Y_{x\phi} = Y(\phi) \times p\ (x) = Y\ p \qquad (99)$$

and,

$$Z_{x\phi} = Z(\phi) \times p\ (x) = Z\ p \qquad (100)$$

With reference to Fig. 3(*a*), the stress components, T_1, T_2, S, are obtained from the equilibrium conditions on a small element. By projection on the z-axis:

$$T_2 = -\ R\ Z\ p \qquad (101)$$

and by projection on the y-axis:

$$\frac{\partial S}{\partial x} = \left(-\ Y + \frac{\partial Z}{\partial \phi}\right) p \qquad (102)$$

Therefore, also,

$$S = \left(-\ Y + \frac{\partial Z}{\partial \phi}\right) \int_0^x p\ dx \qquad (103)$$

By projection on the x-axis:

$$\frac{\partial T_1}{\partial x} + \frac{\partial S}{\partial s} = 0 \qquad (104)$$

and by integration, from Equations (4) and (103):

$$T_1 = \frac{1}{R}\left(\frac{\partial Y}{\partial \phi} + \frac{\partial^2 Z}{\partial \phi^2}\right) M \qquad (105)$$

The membrane stress deflections are obtained by substituting these stress components in Equations (8), (9), and (10). From Equations (8), (101), and (102), by partial integration,

$$E\,t\,u = \frac{1}{R}\left(-\frac{\partial Y}{\partial \phi} + \frac{\partial^2 Z}{\partial \phi^2}\right)\int_{-0.5\,L}^{x} M\,dx + q\,R\,Z\int_{-0.5\,L}^{x} p\,dx \quad \ldots\ldots(106)$$

From Equation (10), by partial differentiation:

$$\frac{\partial^2 v}{\partial x^2} = \frac{2\,(1+m)}{E\,t}\,\frac{\partial S}{\partial x} - \frac{\partial^2 u}{\partial s\,\partial x} \quad \ldots\ldots\ldots\ldots\ldots(107)$$

Substituting, $\frac{\partial S}{\partial x}$, from Equation (102), and, with the partial derivative from Equation (8), in connection with Equations (101) and (105),

$$E\,t\frac{\partial^2 v}{\partial x^2} = 2\,p\left\{-(1+m)\,Y + \left(1+\frac{m}{2}\right)\frac{\partial z}{\partial \phi}\right\} - \frac{M}{R^2}\left\{-\frac{\partial^2 Y}{\partial \phi^2} + \frac{\partial^3 Z}{\partial \phi^3}\right\} \ldots(108)$$

By integration, in connection with Equations (4) and (6),

$$E\,t\,v = 2\,M\left\{(1+m)\,Y - \left(1+\frac{m}{2}\right)\frac{\partial Z}{\partial \phi}\right\} + \frac{\eta}{R^2}\left\{-\frac{\partial^2 Y}{\partial \phi^2} + \frac{\partial^3 Z}{\partial \phi^3}\right\} \ldots(109)$$

From Equation (9,) in connection with Equations (101), (105), and (109),

$$E\,t\,w = R^2 Z\,p + 2\,m\left\{\left(1+\frac{m}{2}\right)\frac{\partial Y}{\partial \phi} - \frac{\partial^2 Z}{\partial \phi^2}\right\} + \frac{\eta}{R^2}\left\{-\frac{\partial^3 Y}{\partial \phi^3} + \frac{\partial^4 Z}{\partial \phi^4}\right\} \ldots(110)$$

The angular deflection of the transverse tangent is negligible in the case of membrane stresses; therefore, $\theta = 0$.

Example

Consider the partial filling of a simply supported, horizontal pipe line or liquid storage tank, provided with adequate transverse stiffening members over both end supports. Neglecting the weight of the shell, the surface load then consists of the liquid pressure, acting continuously over the wetted perimeter of the shell.

At the liquid level the surface load becomes discontinuous, which condition can be analyzed when the shell is cut along this plane and the membrane stress equations are applied to the loaded shell sector. It will be found that a direct stress component, T_1, and a longitudinal shearing stress component, S, are acting on the free edges of the lower shell. The latter force must be applied as a reaction on the edges of the upper shell, where it induces edge load stresses and deformations. The lower shell is deformed by membrane stresses and the combined effect will be a discrepancy in the geometrical continuity of the two shell sectors.

The membrane stress component, S, representing the main disturbing force, reaches a maximum value when the pipe is exactly half full. For this

reason the following derivations will be limited to this loading, which is of practical importance, although any other filling may be analyzed in a similar manner. Referring to Fig. 7, and designating the unit weight of the liquid by q, the external load components are,

$$X = Y = 0 \quad \ldots\ldots (111)$$

and,

$$Z = -\,q\,R \sin \phi \quad \ldots\ldots (112)$$

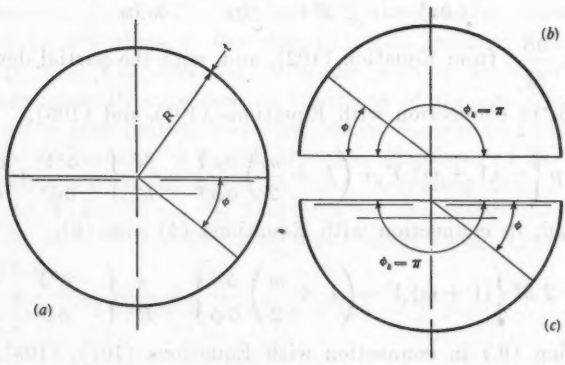

FIG. 7.

The membrane stress components and deformations at $x = 0$, $\phi = 0$, are obtained from Equations (101), (102), (105), (109), and (110):

$$T_2 = 0 \quad \ldots\ldots (113)$$

$$\frac{\partial S}{\partial x} = -\,\frac{4}{c}\,q\,R \quad \ldots\ldots (114)$$

$$T_1 = 0 \quad \ldots\ldots (115)$$

$$E\,t\,v = +\,\frac{4}{c}\left\{2\left(1 + \frac{m}{2}\right) + \frac{1}{a}\right\}\frac{q\,R^3}{a} \quad \ldots\ldots (116)$$

$$E\,t\,w = 0 \quad \ldots\ldots (117)$$

and, since the angular deflection is negligible,

$$\theta = 0 \quad \ldots\ldots (118)$$

Since $T_2 = 0$, the membrane stress component, $\frac{\partial S}{\partial x}$, Equation (114), represents the only reaction that must be applied as a symmetrical edge load on the upper shell. The corresponding deflections and the longitudinal edge stress of the upper shell are obtained from the influence numbers, Table 8

(Appendix II), which already contain the term, $\frac{4}{c}$, as a unit load. Therefore:

$$E\,t\,\theta = +\,c_{14}\,\frac{r^2}{c\,a^2}\,q\,R^2 \ldots\ldots\ldots\ldots\ldots\ldots (119)$$

$$E\,t\,w = -\,c_{24}\,\frac{r^{1.5}}{c\,a^2}\,q\,R^2 \ldots\ldots\ldots\ldots\ldots\ldots (120)$$

$$E\,t\,v = +\,c_{34}\,\frac{r}{c\,a^2}\,q\,R^2 \ldots\ldots\ldots\ldots\ldots\ldots (121)$$

and,

$$T_1 = -\,c_{44}\,\frac{r^{0.5}}{c\,a}\,q\,R^2 \ldots\ldots\ldots\ldots\ldots\ldots (122)$$

A comparison of the deflections, v, in Equations (121) and (116), indicates that the latter is relatively small in the case of thin pipes and, therefore, will be neglected.

A symmetric equilibrium system of unit edge loads, $M_2 = \frac{4}{c}$, $N_2 = \frac{4}{c}$, $T_2 = \frac{4}{c}$, and $\frac{\partial S}{\partial x} = \frac{4}{c}$, is now applied at the edges of both shells. The corresponding deformations, as given by Table 8, are shown in Table 10 (Appendix II), together with the deformations of the membrane stress system, Equations (116) to (122). Table 10 also gives the algebraic differences between the deflections of the upper and lower shell, including the differences between the longitudinal edge stresses. The continuity between the two shells is restored when the differential deformations and stresses are eliminated.

For this purpose four unknown edge force components, designated by $M_2 = X_1$, $N_2 = X_2$, $T_2 = X_3$, and $\frac{\partial S}{\partial x} = X_4$, are applied, so that the algebraic sum of the difference, Δ, becomes zero. Table 10 then gives the following elastic equations for the determination of the unknowns,

$$+\,X_1\,c_{11}\,r^{1.5} - X_3\,c_{13}\,r^{0.5}\,R + c_{14}\,\frac{q\,R^2}{2} = 0 \ldots\ldots\ldots\ldots (123)$$

$$-\,X_2\,c_{22}\,r - X_4\,c_{24}\,R - c_{24}\,\frac{q\,R^2}{2} = 0 \ldots\ldots\ldots\ldots (124)$$

$$+\,X_1\,c_{31}\,r^{1.5} + X_2\,c_{32}\,r\,R - X_3\,c_{33}\,r^{0.5}\,R + c_{34}\,\frac{q\,R^2}{2} = 0 \ldots\ldots (125)$$

and,

$$-\,X_2\,c_{42}\,r - X_4\,c_{44}\,R - c_{44}\,\frac{q\,R^2}{2} = 0 \ldots\ldots\ldots\ldots\ldots (126)$$

The solution of Equations (123) to (126) gives the following values for the unknowns,

$$X_1 = C_2 \frac{q\,R^2}{2\,r^{1.5}} \quad \ldots\ldots (127)$$

$$X_2 = 0 \quad \ldots\ldots (128)$$

$$X_3 = C_1 \frac{q\,R^2}{2\,r^{0.5}} \quad \ldots\ldots (129)$$

and,

$$X_4 = -\frac{q\,R}{2} \quad \ldots\ldots (130)$$

The coefficients, C_1 and C_2 are defined by the following equations,

$$C_1 = \frac{c_{13}\,c_{14} - c_{11}\,c_{34}}{c^2_{13} - c_{11}\,c_{33}} \quad \ldots\ldots (131)$$

and,

$$C_2 = C_1 \frac{c_{33}}{c_{13}} - \frac{c_{34}}{c_{13}} \quad \ldots\ldots (132)$$

The unit stresses at the liquid level, in the center of the span, may be determined by considering the stress condition of the upper shell, which is subject to edge loads only.

The total longitudinal edge stress component consists of the individual components due to the edge loads, X_1 to X_4, and the component of the membrane stress system. The latter is given by Equation (122) and the components caused by X_1, X_3, and X_4, are obtained from the influence numbers, Table 8 (Appendix II). Therefore, the resulting longitudinal edge stress component, designated by T_{1r}, becomes,

$$T_{1r} = -X_1 \frac{c_{41}\,r^2}{R\,c\,a} + X_3 \frac{c_{43}\,r}{c\,a} - X_4 \frac{c_{44}\,r^{0.5}\,R}{c\,a} - c_{44} \frac{r^{0.5}}{c\,a} q\,R^2 \ldots (133)$$

By substituting the corresponding values from Equations (127), (129), and (130), the following expression is obtained,

$$T_{1r} = -C_3 \frac{q\,R^2\,r^{0.5}}{2\,a\,c} \quad \ldots\ldots (134)$$

in which,

$$C_3 = c_{44} - C_1\,c_{34} + C_2\,c_{14} \quad \ldots\ldots (135)$$

The unit stresses, corresponding to the stress components, X_1, X_3 and T_{1r}, in Equations (127), (129), and (134), are designated by σ_{M2}, σ_{T2}, and σ_{T1}, respectively, hence:

$$\sigma_{M2} = \pm \frac{6\,X_1}{t^2} = \pm C_2 \frac{3\,q\,R^2}{t^2\,r^{1.5}} \quad \ldots\ldots (136)$$

$$\sigma_{T2} = C_1 \frac{q\,R^2}{2\,t\,r^{0.5}} \quad \ldots\ldots (137)$$

and,

$$\sigma_{T1} = -\,C_3 \frac{q\,R^2\,r^{0.5}}{2\,a\,c\,t} \qquad \ldots\ldots(138)$$

The value of σ_{T2}, being positive, is a tensile stress; σ_{T1} is a compressive stress. The tangential stress, σ_{T2}, is relatively small. However, it is interesting to compare the transverse bending stress, σ_{M2}, and the longitudinal stress σ_{T1}, with the maximum beam stress of a simply supported, full pipe, designated by σ_B, which is,

$$\sigma_B = \frac{q\,L^2}{8\,t} = \frac{4}{c}\,\frac{q\,L^2}{c^2\,t} \qquad \ldots\ldots(139)$$

Substituting r, from Equation (51), and using a value, $m^2 = 0.10$, for Poisson's ratio:

$$\frac{\sigma_{M2}}{\sigma_B} = 15.9\,C_2 \sqrt{\frac{R}{L}\left(\frac{R}{t}\right)^{\frac{1}{2}}}\,\frac{1}{c^{1.5}} \qquad \ldots\ldots(140)$$

and,

$$\frac{\sigma_{T1}}{\sigma_B} = 4.54\,C_3 \sqrt{\frac{R}{L}\left(\frac{R}{t}\right)^{\frac{1}{2}}}\,\frac{1}{c^{2.5}} \qquad \ldots\ldots(141)$$

Substituting C_2 and C_3, from Equations (132) and (135), respectively, it will be noted that the influence number coefficients are given as functions of $\frac{K_1\,\phi_k}{\pi}$, in Table 9(a) (Appendix II), or Fig. 6(a). Since, in the present example, $\phi_k = \pi$, and K_1, from Table 1 (Appendix II), is also a function of r, it will be possible, therefore, to write Equations (140) and (141) in the following form,

$$\frac{\sigma_{M2}}{\sigma_B} = C_{M2} \sqrt{\frac{R}{L}\left(\frac{R}{t}\right)^{\frac{1}{2}}} \qquad \ldots\ldots(142)$$

and,

$$\frac{\sigma_{T1}}{\sigma_B} = C_{T1} \sqrt{\frac{R}{L}\left(\frac{R}{t}\right)^{\frac{1}{2}}} \qquad \ldots\ldots(143)$$

The values of the coefficients, C_{M2} and C_{T1}, have been computed for the first term of the Fourier series, by substituting, $c = \pi$, and are shown in Fig. 8. As already stated, the consideration of additional terms of the series does not greatly influence the final result. Due to the approximations made in the derivation of the influence numbers it is doubtful whether the accuracy would be increased thereby.

By an entirely different method, Samsioe[12] has analyzed the same problem for a pipe half full, but with fixed ends, instead of the simply supported conditions assumed in the present example. Although the results for this reason are not directly comparable, the corresponding coefficients, derived from Samsioe's analysis, are also shown in Fig. 8.

[12] "Die Spannungen in einem auf mehreren Stützen in gleicher gegenseitiger Entfernung aufgelegten und zur Hälfte mit Wasser gefüllten Rohr," von A. Frey Samsioe, pub. by Svenska Bokhandelscentralen, Stockholm, Sweden, 1926.

It will be noted that the maximum stresses due to the discontinuous surface load are of similar magnitude in both cases, except that the limitations

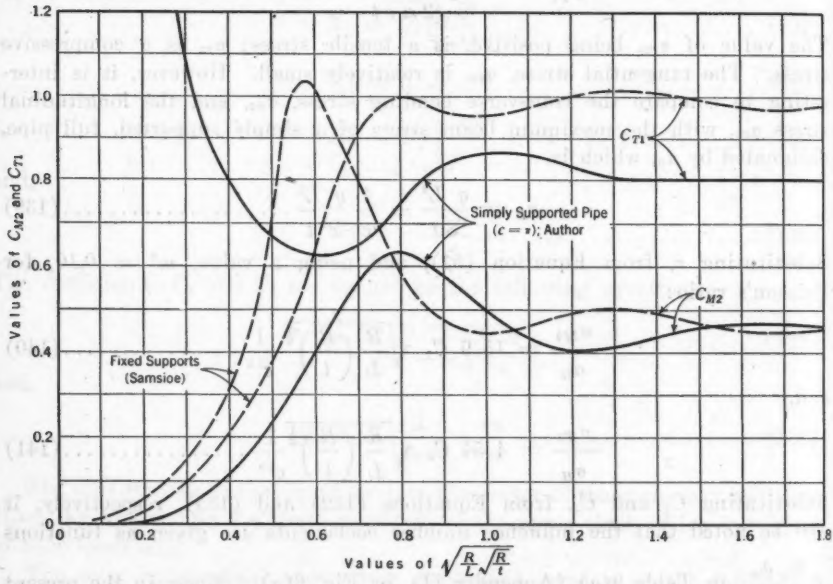

FIG. 8.—COEFFICIENT C FOR A PIPE HALF FULL OF WATER, WITH ANGLE SUPPORTS AND FIXED SUPPORTS.

of the present method are indicated in case of small values of $\sqrt{\frac{R}{L}\left(\frac{R}{t}\right)^{\frac{1}{2}}}$.

CONCLUSION

Thin shell structures will find an increasingly large field of useful application because of their economic advantages and their adaptability to architectural treatment in wide span construction. For practical purposes the analysis of such structures must be simplified so as to enable the designer to obtain a reasonably accurate stress picture in spite of the fact that exact solutions are not yet available on account of mathematical difficulties.

The proposed method is an approximation, but the comparison of the results with more accurate derivations indicates a satisfactory agreement, provided the given limitations are kept in mind.

APPENDIX I

NOTATION

The symbols introduced in this paper are defined where they first appear; and, for the convenience of discussers (who are requested to adapt their

remarks to it) the complete notation, in alphabetical order, is listed herein, as follows:

a = a factor, $\left(\frac{c\,R}{L}\right)^2$; also, a^*, b^*, c^*, and d^* = functions of the angle, ϕ; and, a_o, b_o, c_o, and d_o = edge values for $\phi = 0$;

b = a shell constant $= \frac{R}{T}\sqrt{3}$; also, for b^* and b_o, refer to Symbol a;

c = substitutions corresponding to the first, second, third, etc., terms of a Fourier series; c_{11}, c_{12}, c_{13}, etc. = influence number coefficients for edge deflections; for c^* and c_o refer to Symbol a;

d^* = function of the angle, ϕ (see Symbol a).

e = base of natural logarithms; also e = elongation: e_1, longitudinal, and e_2, transverse;

f = "function of": $f(\phi)$ = Finsterwalder's stress function for a cylindrical shell sector; f, g, and h = functions of J_1, K_1, and ϕ_k; f', g', and h' = functions of J', K', and ϕ'_k;

g = "function of": (for g and g', refer to Symbol f);

h = "function of": (for h and h', refer to Symbol f);

k = a constant $= \sqrt{2} - 1$;

m = Poisson's ratio;

n = a constant: $n_1 = \sqrt{0.5} + 0.5$ and $n_2 = \sqrt{0.5} - 0.5$;

p = load intensity per unit length of a simply supported beam; $p(x)$ = sum of a trigonometric series; p_c = general term of series;

q = unit weight of liquid;

r = a shell constant $= \sqrt{a, b}\sqrt[4]{1 - m^2}$;

s = length of a circular arc corresponding to a central angle, ϕ; ds = length of a small element of arc, s;

t = thickness of shell; as a subscript, t, denotes "torsional";

u = longitudinal deflection of a shell element, positive in the direction of increasing values of x;

v = tangential deflection of a shell element, positive in the direction of increasing values of ϕ;

w = radial deflection of a shell element, positive in a direction toward the center of curvature;

x = a distance measured longitudinally along the arch barrel, from the center of the span; dx = an elementary small distance, x;

A = a constant; A, B, C, and D = constants conforming to given edge conditions of the shell;

B = a constant (see Symbol A); as a subscript, B, denotes "beam";

C = a constant (see Symbol A); C_1, C_2, and C_3 = functions of the influence number coefficients referred to in the example in this paper;

D = a constant (see Symbol A);

E = modulus of elasticity;

F = a function; F and G = functions determining J and K;

G = a function (see Symbol F);

I = rectangular moment of inertia of a beam section;

J = a shell constant; J_1 = a damping factor (as an exponent, written simply, $(J1)$); J'_1 = a damping factor of stress functions (as an exponent, written, $(J1)'$);

K = a shell constant; K_1 and K'_1 = wave length factors of stress function; K_n = a progressive function of K_1;

L = length; span of a simply supported beam or cylindrical shell;

M = moment; bending moment of a simply supported, uniformly loaded beam at the distance, x; M_0 = bending moment at the center of the span; M_1 = longitudinal bending stress component on a shell element per unit width (positive when tension is produced on the inner surface); M_2 = transverse bending moment corresponding to M_1; M_t = torsional moment acting on a shell element per unit width (positive when producing tension on the outer surface, in a diagonal direction of increasing values of x and ϕ); M_c = general term of a series;

N = radial shearing stress component per unit width (positive when acting in an outward direction on the sides facing the origin of x and ϕ);

R = radius of circular shell; radius of curvature of a cylinder element;

S = direct shearing stress component on a shell element per unit width, acting in the plane of the element (positive when producing tension in a diagonal direction of increasing values of x and ϕ);

T = direct stress component on a shell element per unit width; T_1, in a longitudinal direction, and T_2 in a tangential, or transverse, direction; T_{1r} = resulting longitudinal edge stress component in the design example of the paper;

X = force component: $X_{x\phi}$, $Y_{x\phi}$, and $Z_{x\phi}$ = external load components on a shell element per unit width; X_1, X_2, X_3, and X_4 = unknown edge force components in the design example of the paper;

Y = force component (see Symbol X);

Z = force component (see Symbol X);

α = a constant in Table 2 (Appendix II);

β = a constant in Table 2 (Appendix II);

γ = a constant in Table 2 (Appendix II);

δ = a constant in Table 2 (Appendix II);

Δ = a correction, or difference; ΔJ_2 = correction for J_2; ΔK_2 = correction for K_2; Δ_1 = errors in values of J; Δ_2 = errors in values of K;

η = deflection of a simply supported, uniformly loaded beam, multiplied by EI; $\eta_0 = \eta$ at the center of the span; η_c = general term of a series;

θ = angular deflection of the transverse tangent of a shell element (positive when the tangent rotates in a clockwise direction);

κ = change of curvature of a shell element: κ_1 = longitudinal, and κ_2 = transverse;

λ = a typical shell constant defined as a ratio $= \sqrt{r}\,\phi_k$;

σ = unit stress; σ_B = maximum beam stress in a simply supported full pipe; σ_{M2}, σ_{T2}, and σ_{T1} = unit stresses corresponding to stress components, X_1, X_2, and T_{1r}, respectively;

ϕ = angular distance of a shell element from the left-end edge; $d\phi$ = angular width of a small shell element; ϕ_k = total central angle of a cylindrical shell sector; $f(\phi)$ = Finsterwalder's stress function for a cylindrical shell sector;

ω = angular detrusion of a shell element, caused by the shearing forces, S (positive when S is positive).

APPENDIX II

TABLES

The ten tables in Appendix II are introduced and explained in the text of the paper.

TABLE 1.—APPROXIMATE VALUES OF J, K, J', AND K'

Values	Indices							
	1	2	3	4	5	6	7	8
J	$-n_1 r^{0.5}$	$+r$	$-(n_1-n_2) r^{1.5}$	0	$+2 n_2 r^{2.5}$	$-2r^2$	$2(n_1+n_2) r^{3.5}$	$-4r^4$
K	$+n_2 r^{0.5}$	$-r$	$+(n_1+n_2) r^{1.5}$	$-2r^2$	$+2 n_1 r^{2.5}$	$-2r^2$	$2(n_1-n_2) r^{3.5}$	0
J'	$-n_2 r^{0.5}$	$-r$	$+(n_1+n_2) r^{1.5}$	0	$-2 n_1 r^{2.5}$	$+2r^2$	$2(n_1-n_2) r^{3.5}$	$-4r^4$
K'	$+n_1 r^{0.5}$	$-r$	$-(n_1-n_2) r^{1.5}$	$+2r^2$	$-2 n_2 r^{2.5}$	$-2r^2$	$2(n_1+n_2) r^{3.5}$	0

TABLE 2.—CONSTANTS α, β, γ, AND δ

Stress component or deflection	α	β	γ	δ	Common factor	Stress component or deflection	α	β	γ	δ	Common factor
(1)	(2)	(3)	(4)	(5)	(6)	(1)	(2)	(3)	(4)	(5)	(6)
M_2	-1	0	-1	0	1	T_1	0	$+1$	0	-1	$\frac{2r^2}{aR}$
N_2	$+1$	$-k$	$+k$	-1	$\frac{n_1 r^{0.5}}{R}$	v	$-k$	-1	$+1$	$+k$	$\frac{2 n_1 r^{2.5}}{E t a^2}$
T_2	$+1$	-1	-1	-1	$\frac{r}{R}$	w	$+1$	$+1$	-1	$+1$	$\frac{2r^3}{E t a^2}$
$\frac{\partial S}{\partial x}$	$+k$	-1	-1	$+k$	$\frac{(n_1+n_2) r^{1.5}}{R^2}$	θ	-1	$-k$	$-k$	-1	$\frac{2(n_1+n_2) r^{2.5}}{E t R a^2}$

TABLE 3.—EDGE VALUES, a_0, b_0, c_0, and d_0 (AT THE EDGE, $\phi = 0$)

Stress component or deflection	a_0	b_0	c_0	d_0	Common factor
			(a) SYMMETRICAL LOADING		
M_2.....	$-f$	$-g$	$-f'$	$-g'$	1
N_2.....	$-(k\,h+f)$	$+(h-k\,f)$	$-(h'+k\,f')$	$+(k\,h'-f')$	$\frac{n_1\,r^{0.5}}{R}$
T_2.....	$-(g-f)$	$+(g-f)$	$-(g'+f')$	$-(g'-f')$	$\frac{r}{R}$
$\frac{\partial S}{\partial x}$	$-(h+k\,f)$	$+(k\,h-f)$	$+(k\,h'+f')$	$-(h'-k\,f')$	$\frac{(n_1+n_2)\,r^{1.5}}{R^2}$
T_1.....	$+g$	$-f$	$-g'$	$+f'$	$\frac{2\,r^2}{a\,R}$
v.....	$-(h-k\,f)$	$-(k\,h+f)$	$+(k\,h'-f')$	$+(h'+k\,f')$	$\frac{2\,n_1\,r^{2.5}}{E\,t\,a^2}$
w	$+(g+f)$	$+(g-f)$	$+(g'-f')$	$-(g'+f')$	$\frac{2\,r^3}{E\,t\,a^2}$
θ	$-(k\,h-f)$	$-(h+k\,f)$	$-(h'-k\,f')$	$-(k\,h'+f')$	$\frac{2\,(n_1+n_2)\,r^{2.5}}{E\,t\,R\,a^2}$
			(b) CONTRA-SYMMETRICAL LOADING		
(1) M_2....	$+f$	$-h$	$+f'$	$-h'$	1
(2) N_2.....	$-(k\,g-f)$	$+(g+k\,f)$	$-(g'-k\,f')$	$+(k\,g'+f')$	$\frac{n_1\,r^{0.5}}{R}$
(3) T_2.....	$-(h+f)$	$+(h-f)$	$-(h'-f')$	$-(h'+f')$	$\frac{r}{R}$
(4) $\frac{\partial S}{\partial x}$ ...	$-(g-k\,f)$	$+(k\,g+f)$	$+(k\,g'-f')$	$-(g'+k\,f')$	$\frac{(n_1+n_2)\,r^{1.5}}{R^2}$
(5) T_1.....	$+h$	$+f$	$-h'$	$-f'$	$\frac{2\,r^2}{a\,R}$
(6) v.....	$-(g+k\,f)$	$-(k\,g-f)$	$+(k\,g'+f')$	$+(g'-k\,f')$	$\frac{2\,n_1\,r^{2.5}}{E\,t\,a^2}$
(7) w.....	$+(h-f)$	$+(h+f)$	$+(h'+f')$	$-(h'-f')$	$\frac{2\,r^3}{E\,t\,a^2}$
(8) θ.....	$-(k\,g+f)$	$-(g-k\,f)$	$-(g'+k\,f')$	$-(k\,g'-f')$	$\frac{2\,(n_1+n_2)\,r^{2.5}}{E\,t\,R\,a^2}$

TABLE 4.—CONSTANTS A, B, C, AND D ($M_2 = \frac{4}{c}$, $N_2 = T_2 = \frac{\partial S}{\partial x} = 0$)

$\frac{K_1 \phi_k}{\pi}$	A_c	B_c	C_c	D_c
	(a) SYMMETRICAL LOADING			
0.000	.....	.....	.....	.....
0.125	−0.289	−1.559	−0.548	−1.104
0.250	−0.817	−2.306	−1.217	−0.978
0.375	−1.681	−3.272	−1.953	−0.550
0.500	−2.978	−4.474	−2.679	+0.245
0.625	−4.748	−5.867	−3.273	+1.421
0.750	−6.703	−7.167	−3.533	+2.762
0.875	−7.940	−7.768	−3.319	+3.619
1.000	−7.821	−7.453	−2.901	+3.529
1.125	−7.096	−6.901	−2.685	+3.017
1.250	−6.634	−6.641	−2.703	+2.690
1.375	.....	.....	.....	.....
1.500	−6.712	−6.777	−2.855	+2.744
1.625	.....	.....	.....	.....
1.750	.....	.....	.....	.....
1.875	.....	.....	.....	.....
2.000	.....	.....	.....	.....
∞	−6.828	−6.828	−2.828	+2.828
	(b) CONTRA-SYMMETRICAL LOADING			
0.000	.....	.....	.....	.....
0.125	−186016	−289792	−270220	+16907
0.250	−2054	−2371	−1854	+957
0.375	−199.8	−184.7	−109.2	+120.5
0.500	−41.99	−33.79	−13.43	+26.20
0.625	−14.30	−11.51	−3.333	+7.936
0.750	−7.570	−6.946	−2.287	+3.327
0.875	−5.996	−6.154	−2.510	+2.237
1.000	−6.072	−6.383	−2.825	+2.295
1.125	−6.622	−6.799	−2.985	+2.684
1.250	−7.042	−7.035	−2.967	+2.980
1.375	.....	.....	.....	.....
1.500	−6.947	−6.879	−2.797	+2.912
1.625	.....	.....	.....	.....
1.750	.....	.....	.....	.....
1.875	.....	.....	.....	.....
2.000	.....	.....	.....	.....
∞	−6.828	−6.828	−2.828	+2.828

TABLE 5.—CONSTANTS A, B, C, AND D ($N_2 = \frac{4}{c}$, $M_2 = T_2 \frac{\partial S}{\partial x} = 0$)

$\frac{K_1 \phi_k}{\pi}$	$A \frac{c r^{0.5}}{R}$	$B \frac{c r^{0.5}}{R}$	$C \frac{c r^{0.5}}{R}$	$D \frac{c r^{0.5}}{R}$
	(a) SYMMETRICAL LOADING			
0.000	.....	.....	.....	.....
0.125	−1348	−959	−450	+1255
0.250	−67.40	−36.32	−0.033	+55.42
0.375	−14.21	−6.288	+1.017	+10.06
0.500	−7.159	−3.654	−1.323	+4.372
0.625	−7.529	−4.906	−3.056	+4.504
0.750	−10.44	−7.232	−4.104	+6.511
0.875	−13.33	−8.951	−4.145	+8.526
1.000	−13.94	−8.914	−3.468	+8.951
1.125	−12.85	−7.958	−2.936	+8.174
1.250	−11.80	−7.289	−2.843	+7.431
1.375	.....	.....	.....	.....
1.500	−11.55	−7.308	−3.114	+7.257
1.625	.....	.....	.....	.....
1.750	.....	.....	.....	.....
1.875	.....	.....	.....	.....
2.000	.....	.....	.....	.....
∞	−11.90	−7.502	−3.108	+7.502

TABLE 5.—(*Continued*)

$\frac{K_1 \phi_k}{\pi}$	$A\frac{c\,r^{0.5}}{R}$	$B\frac{c\,r^{0.5}}{R}$	$C\frac{c\,r^{0.5}}{R}$	$D\frac{c\,r^{0.5}}{R}$
	(*b*) CONTRA-SYMMETRICAL LOADING			
0.000	.....	.....	.....	.....
0.125	−80256	−125028	−116585	+7295
0.250	−1773	−2044	−1600	+827
0.375	−258.3	−236.0	−141.1	+157.7
0.500	−71.36	−53.70	−22.05	+47.08
0.625	−28.49	−18.30	−4.901	+18.89
0.750	−15.31	−8.982	−2.217	+9.858
0.875	−11.08	−6.551	−2.297	+6.919
1.000	−10.38	−6.474	−2.866	+6.432
1.125	−11.13	−7.161	−3.293	+6.960
1.250	−12.04	−7.752	−3.388	+7.607
1.375	.....	.....	.....	.....
1.500	−12.23	−7.687	−3.089	+7.738
1.625	.....	.....	.....	.....
1.750	.....	.....	.....	.....
1.875	.....	.....	.....	.....
2.000	.....	.....	.....	.....
∞	−11.90	−7.502	−3.108	+7.502

TABLE 6.—CONSTANTS A, B, C, AND D ($T_2 = \frac{4}{c}$, $M_2 = N_2 = \frac{\partial S}{\partial x} = 0$)

$\frac{K_1 \phi_k}{\pi}$	$A\frac{c\,r}{R}$	$B\frac{c\,r}{R}$	$C\frac{c\,r}{R}$	$D\frac{c\,r}{R}$
	(*a*) SYMMETRICAL LOADING			
0.000	.....	.....	.....	.....
0.125	+581	+414	+193	−542
0.250	+57.15	+32.17	−1.458	−48.02
0.375	+16.39	+8.264	−3.969	−12.70
0.500	+7.784	+4.327	−2.346	−5.748
0.625	+6.162	+4.300	+0.811	−4.432
0.750	+7.569	+5.816	+0.361	−5.379
0.875	+10.01	+7.495	+0.787	−7.083
1.000	+11.17	+7.973	+0.442	−7.897
1.125	+10.69	+7.421	−0.021	−7.559
1.250	+9.839	+6.820	−0.196	−6.956
1.375	.....	.....	.....	.....
1.500	+9.318	+6.614	−0.035	−6.592
1.625	.....	.....	.....	.....
1.750	.....	.....	.....	.....
1.875	.....	.....	.....	.....
2.000	.....	.....	.....	.....
∞	+9.657	+6.741	0.000	−6.741
	(*b*) CONTRA-SYMMETRICAL LOADING			
0.000	.....	.....	.....	.....
0.125	+13878	+21592	+20113	−1274
0.250	+615	+707	+549	−288
0.375	+134.4	+123.1	+70.91	−82.50
0.500	+48.97	+37.80	+13.24	−32.92
0.625	+23.51	+16.13	+1.992	−16.34
0.750	+13.77	+8.964	−0.472	−9.688
0.875	+9.802	+6.488	−0.724	−6.923
1.000	+8.579	+5.978	−0.373	−6.066
1.125	+8.805	+6.340	+0.018	−6.227
1.250	+9.516	+6.860	+0.197	−6.729
1.375	.....	.....	.....	.....
1.500	+9.973	+7.028	+0.025	−7.052
1.625	.....	.....	.....	.....
1.750	.....	.....	.....	.....
1.875	.....	.....	.....	.....
2.000	.....	.....	.....	.....
∞	+9.657	+6.741	0.000	−6.741

TABLE 7.—CONSTANTS A, B, C, AND D $\left(\frac{\partial S}{\partial x} = \frac{4}{c}, M_2 = N_2 = T_2 = 0\right)$

$\frac{K_1 \phi_k}{\pi}$	$A\frac{c r^{1.5}}{R^2}$	$B\frac{c r^{1.5}}{R^2}$	$C\frac{c r^{1.5}}{R^2}$	$D\frac{c r^{1.5}}{R^2}$
	(a) SYMMETRICAL LOADING			
0.000	.....	.....	.....	.....
0.125	−85.34	−59.33	−26.50	+77.30
0.250	−17.70	−8.861	+1.101	+13.12
0.375	−8.120	−2.998	+2.212	+4.676
0.500	−5.168	−1.514	+1.906	+2.241
0.625	−4.187	−1.227	+1.439	+1.453
0.750	−4.314	−1.573	+0.983	+1.513
0.875	−5.080	−2.182	+0.712	+2.043
1.000	−5.669	−2.515	+0.731	+2.455
1.125	−5.690	−2.445	+0.873	+2.470
1.250	−5.452	−2.255	+0.959	+2.301
1.375	.....	.....	.....	.....
1.500	−5.189	−2.117	+0.935	+2.117
1.625	.....	.....	.....	.....
1.750	.....	.....	.....	.....
1.875	.....	.....	.....	.....
2.000	.....	.....	.....	.....
∞	−5.305	−2.197	+0.910	+2.197
	(b) CONTRA-SYMMETRICAL LOADING			
0.000	.....	.....	.....	.....
0.125	−1005	−1553	−1439	+96.15
0.250	−91.84	−101.9	−76.78	+42.05
0.375	−31.18	−26.46	−13.96	+17.72
0.500	−15.79	−10.59	−2.829	+9.090
0.625	−9.866	−5.361	+0.090	+5.307
0.750	−7.068	−3.209	+0.949	+3.418
0.875	−5.665	−2.274	+1.139	+2.446
1.000	−5.061	−1.955	+1.080	+2.024
1.125	−4.982	−1.983	+0.953	+1.969
1.250	−5.171	−2.145	+0.866	+2.103
1.375	.....	.....	.....	.....
1.500	−5.410	−2.272	+0.889	+2.272
1.625	.....	.....	.....	.....
1.750	.....	.....	.....	.....
1.875	.....	.....	.....	.....
2.000	.....	.....	.....	.....
∞	−5.305	−2.197	+0.910	+2.197

TABLE 8.—INFLUENCE NUMBERS FOR EDGE DEFLECTIONS AT $x = 0$, $\phi = 0$

Item No.	LOAD CONDITION				EDGE DEFLECTIONS			
	M_2	N_2	T_2	$\frac{\partial S}{\partial x}$	θ (1)	w (2)	v (3)	T_1 (4)
(1)	$\frac{4}{c}$	0	0	0	$+\frac{c_{11} r^{2.5}}{E t R c a^2}$	$-\frac{c_{21} r^3}{E t c a^2}$	$+\frac{c_{31} r^{2.5}}{E t c a^2}$	$-\frac{c_{41} r^2}{R c a}$
(2)	0	$\frac{4}{c}$	0	0	$+\frac{c_{12} r^3}{E t c a^2}$	$-\frac{c_{22} R r^{2.5}}{E t c a^2}$	$+\frac{c_{32} R r^3}{E t c a^2}$	$-\frac{c_{42} r^{1.5}}{c a}$
(3)	0	0	$\frac{4}{c}$	0	$-\frac{c_{13} r^{2.5}}{E t c a^2}$	$+\frac{c_{23} R r^2}{E t c a^2}$	$-\frac{c_{33} R r^{1.5}}{E t c a^2}$	$+\frac{c_{43} r}{c a}$
(4)	0	0	0	$\frac{4}{c}$	$+\frac{c_{14} R r^2}{E t c a^2}$	$-\frac{c_{24} R^2 r^{1.5}}{E t c a^2}$	$+\frac{c_{34} R^2 r}{E t c a^2}$	$-\frac{c_{44} r^{1.5} R}{c a}$

TABLE 9.—Influence Number Coefficients

$\frac{K_1 \phi_b}{\pi}$	c_{11}	c_{12} and c_{21}	c_{13} and c_{31}	c_{14} and c_{41}	c_{22}	c_{23} and c_{32}	c_{24} and c_{42}	c_{33}	c_{34} and c_{43}	c_{44}
				(*a*) Symmetrical Loading						
0.000	0	0	0	0	.....	.....	.....	.....	.....	.....
0.125	6.90	1.28	0.12	0.01	.....	5185	746	2237	322	55.54
0.250	13.81	5.11	0.98	0.08	378.7	325.20	93.44	280.3	80.60	27.82
0.375	20.70	11.50	3.32	0.43	57.51	66.60	28.00	83.80	35.92	18.56
0.500	27.55	20.33	7.79	1.34	30.52	28.02	13.05	38.30	20.74	14.01
0.625	33.88	31.08	14.85	3.20	39.60	26.68	10.06	27.66	15.09	11.62
0.750	38.48	41.35	23.41	6.01	57.92	38.56	12.64	32.28	14.87	10.88
0.875	39.37	46.10	29.42	8.66	71.57	51.93	17.45	42.99	18.00	11.56
1.000	37.05	43.59	29.44	9.47	71.27	55.75	20.11	48.91	20.77	12.61
1.125	34.84	39.20	26.33	8.78	63.65	51.36	19.56	47.34	21.08	12.98
1.250	34.22	37.12	24.13	7.97	58.30	46.55	18.04	43.60	20.08	12.79
1.375	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
1.500	35.14	38.22	24.25	7.74	58.46	44.98	16.97	40.92	18.82	12.29
1.625	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
1.750	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
1.875	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
2.000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
∞	35.16	38.63	24.86	8.00	60.02	46.63	17.58	42.44	19.31	12.43
				(*b*) Contra-Symmetrical Loading						
0.000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
0.125	.....	.....	.....	.....	.....	.....	6788	.....	1236	112.1
0.250	.....	.....	5244	754.3	.....	4525	651	1581	240.7	46.20
0.375	1047	1343	693.9	149.6	1735	897.4	193.5	470.0	107.3	30.88
0.500	155.3	246.8	166.8	47.60	416.7	285.6	81.90	198.9	60.49	23.19
0.625	50.33	75.50	58.17	20.02	145.0	120.0	42.39	102.9	38.89	18.57
0.750	33.63	39.56	28.95	10.64	71.73	63.62	25.60	62.15	27.49	15.57
0.875	32.26	33.27	21.37	7.39	52.43	43.93	18.19	44.13	21.30	13.60
1.000	33.98	35.08	21.30	6.76	51.82	39.85	15.65	37.91	18.41	12.43
1.125	35.64	38.41	23.65	7.29	57.30	42.73	15.87	38.40	17.86	12.00
1.250	36.19	40.28	25.65	8.02	62.03	46.90	17.16	41.48	18.61	12.10
1.375	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
1.500	35.17	39.03	25.46	8.27	61.50	48.23	18.17	43.89	19.78	12.55
1.625	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
1.750	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
1.875	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
2.000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
∞	35.16	38.63	24.86	8.00	60.02	46.63	17.58	42.44	19.31	12.43

TABLE 10.—Edge Deformations of Upper and Lower Shell

Item No.	Edge deformations	Load Condition					Common factor
		$M_2 = \frac{4}{c}$	$N_2 = \frac{4}{c}$	$T_2 = \frac{4}{c}$	$\frac{\partial S}{\partial x} = \frac{4}{c}$	Membrane system	
1	θ, upper shell..	$+ c_{11} r^{1.5}$	$+ c_{12} r R$	$- c_{13} r^{0.5} R$	$+ c_{14} R^2$	$+ c_{14} q R^2$	$\frac{E t R c a^2}{r^2}$
2	θ, lower shell..	$- c_{11} r^{1.5}$	$+ c_{12} r R$	$+ c_{13} r^{0.5} R$	$+ c_{14} R^2$	0	
3	$\Delta\theta$ = Item No. 1. — Item No. 2.	$+ 2 c_{11} r^{1.5}$	0	$- 2 c_{13} r^{0.5} R$	0	$+ c_{14} q R^2$	
4	w, upper shell..	$- c_{21} r^{1.5}$	$- c_{22} r R$	$+ c_{23} r^{0.5} R$	$- c_{24} R^2$	$+ c_{24} q R^2$	$\frac{E t c a^2}{r^{1.5}}$
5	w, lower shell..	$- c_{21} r^{1.5}$	$+ c_{22} r R$	$+ c_{23} r^{0.5} R$	$+ c_{24} R^2$	0	
6	Δw = Item No. 4. — Item No. 5.	0	$- 2 c_{22} r R$	0	$- 2 c_{24} R^2$	$- c_{24} q R^2$	
7	v, upper shell .	$+ c_{31} r^{1.5}$	$+ c_{32} r R$	$- c_{33} r^{0.5} R$	$+ c_{34} R^2$	$+ c_{34} q R^2$	$\frac{E t c a^2}{r}$
8	v, lower shell...	$- c_{31} r^{1.5}$	$- c_{32} r R$	$+ c_{33} r^{0.5} R$	$+ c_{34} R^2$	0	
9	Δv = Item No. 7. — Item No. 8.	$+ 2 c_{31} r^{1.5}$	$+ 2 c_{32} r R$	$- 2 c_{33} r^{0.5} R$	0	$+ c_{34} q R^2$	
10	T_1, upper shell.	$- c_{41} r^{1.5}$	$- c_{42} r R$	$+ c_{43} r^{0.5} R$	$- c_{44} R^2$	$- c_{44} q R^2$	$\frac{c a R}{r^{0.5}}$
11	T_1, lower shell.	$- c_{41} r^{1.5}$	$+ c_{42} r R$	$+ c_{43} r^{0.5} R$	$+ c_{44} R^2$	0	
12	ΔT_1 = Item No. 10 — Item No. 11	0	$- 2 c_{42} r R$	0	$- 2 c_{44} R^2$	$- c_{44} q R^2$	

AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

UNDERGROUND CORROSION

BY K. H. LOGAN,[1] ESQ.

SYNOPSIS

Underground corrosion is shown to be of economic importance by the fact that associations of pipe owners and pipe-owning corporations are spending considerable sums for the study of corrosion problems. The underground pipes in the United States have been estimated to have a length of 450 000 miles and a value of nearly $6 000 000 000. The annual loss due to underground corrosion of pipe lines in the oil industry alone has been estimated at $25 000 000.

Soil corrosion is characterized by the uneven distribution of the corrosion and the fact that most ferrous pipe materials corrode at nearly the same rates. The major cause of the corrosion seems to be non-uniformity in the distribution, of oxygen and moisture along the surface of the pipe line.

Among the methods suggested for reducing corrosion losses are the use of copper and copper alloy pipe which corrode less rapidly than ferrous materials; increasing the thickness of the pipe walls; and the use of protective coatings.

Soil corrosivity surveys are suggested as a means for determining the need for protective coatings. Some of the weaknesses of coatings are noted and cathodic protection is suggested as a means of improving the effectiveness of coatings. In an Appendix are listed some of the more important recent articles on underground corrosion and related subjects.

THE IMPORTANCE OF THE PROBLEM

A recent compilation gives the amount of underground trunk and distribution mains in the United States as 450 800 miles, having an estimated value of nearly $6 000 000 000. In addition, there are approximately 14 000 000

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gas-service pipes and probably at least an equal number of water services. No estimate has been made of the mileage of underground pipe in use for waste disposal, but it will approximate that used for water services. The amount and value of the ferrous material exposed to the action of soil, therefore, are enormous, and even a moderate reduction in the rate of deterioration of underground pipe lines is important.

In 1931,[2] Carl R. Weidner, Assoc. M. Am. Soc. C. E., estimated the annual loss to the pipe industry on account of underground corrosion, at $50 000 000. Presumably, this value refers to oil lines, and if so, it should be multiplied by at least three to represent the losses on all underground pipe. In view of the large losses sustained by gas and oil companies on account of underground corrosion and the efforts which these organizations are making to reduce these losses, it is a little surprising that equal attention to the subject is not given by the owners of lines carrying water. Aside from the unlikely explanation that water lines do not corrode, several others might be offered. In general, the owners of water pipes are not as well organized to fight corrosion since they consist largely of independent municipalities and individuals. Water mains usually have greater wall thickness than gas mains of the same diameter and, as will be shown later, wall thickness is an important factor in pipe life. Moreover, water leaks of the same magnitude are less serious than leaks of oil or gas because of the damage which the latter cause to property and because of the fire and explosion hazard accompanying gas leaks in cities. The tendency to use thinner-walled pipe and to place expensive pavement over distribution mains makes it more important to reduce the corrosion losses to a minimum.

At first, it was customary to attribute all underground corrosion in cities to stray-current electrolysis. The improvement of operating conditions on some street railways and the abandonment of street-car lines have focused more attention on soil conditions.

Characteristics and Causes of Underground Corrosion

The findings of the National Bureau of Standards with respect to the action of soils have been presented in various ways and on numerous occasions. The repetition of four major conclusions herein is not for the purpose of imparting new information, but to establish a basis for the discussion of the reasons for the suggestions made in this paper.

Briefly, the Bureau has found: First, that the differences in the rates of corrosion of the commonly used ferrous pipe materials are of the same order of magnitude as those of different specimens of the same material; and, second, that the character and seriousness of soil action vary with the soil condition to which the material is exposed. This latter variation is so great that it appears to be a much more important factor than the choice of ferrous pipe material. Figs. 1, 2, and 3, are views of 6-in. lengths of $1\frac{1}{2}$-in. pipe of three different materials. (If the reader will face toward a source of direct

[2] Opening Remarks to Group Session on "Corrosion of Oil Field Equipment," by C. R. Weidner, *Proceedings*, Am. Petroleum Inst., Vol. 12 Pt. 4, p. 26 (*Production Bulletin 308*).

light, preferably sunlight, so that the captions of Figs. 1, 2, and 3 are on his right, the pitted condition of the pipe surface will be apparent. If the illustrations are reversed the pits will appear as projections.) The similarity of the corrosion pattern of the three materials in the same soil is shown, as is the difference between the patterns in different soils. The wide difference which may occur in the rates of corrosion of the specimens of the same material in the same soil is illustrated by Fig. 1. The third characteristic of soil corrosion which must be considered is that the rate of corrosion after a pipe has been exposed to soil for several years is considerably less than for the first period of exposure. A fourth characteristic of underground corrosion (which not only is of great economic importance, but is suggestive of the cause of corrosion) is the unevenness of the distribution of the corrosion over the surface of the buried metal. Sometimes a length or section of pipe is penetrated at only one point, whereas almost no corrosion is found elsewhere on the section. The pitting is usually confined to the bottom of the pipe, but occasionally the most severe corrosion is found on the sides or top.

The most interesting characteristic of underground corrosion is the irregular or spotty nature of the attack. The loss of material as such is usually too small to be of importance if it were uniformly distributed over the pipe surface. It will be helpful, in understanding these findings, to construct at least a tentative explanation of underground corrosion processes.

When the electrolytic theory of corrosion became generally accepted, the first explanation offered for the formation of pits in the pipe surface was impurities or segregations in the pipe material. This conclusion was supported by the fact that where impurities were known to exist, differences in electrical potential between the impurities and the metal could be measured. It was shown that there was a difference of potential between oxidized metal or mill scale and bright iron, and strains in metal were also shown to result in differences of potential. It was logical to assume that these potential differences resulted in corroding currents when the metal was exposed to an electrolyte. Of course, other differences of potential were observed when different metals, or two varieties of the same metal, were in metallic contact.

The mistake that has been made is not in reasoning but in evaluating the importance of the conclusion. There can be little doubt that some corrosion results as suggested, and that under some conditions one of these causes of corrosion is the controlling one. When materials are buried in soil, however, and under some other conditions as well, other causes of corrosion are more important.

The influence on corrosion of causes external to the pipe was brought forcibly to the attention of the observers at the Bureau of Standards when the first specimens of pipe were examined after approximately ten years of exposure to widely different soils, and is illustrated in Figs. 1, 2, and 3. It was observed that in certain soils the corrosion took the form of isolated pits with sharp margins. In other soils the corroded areas took the form of rather large shallow depressions with no well-defined boundaries. In one soil the removal of metal was very uniform over the entire exposed surface. Differ-

FIG. 1.—SPECIMENS OF OPEN-HEARTH STEEL PIPE CORRODED BY EIGHT YEARS OF CONTACT WITH: (*a*) AND (*d*) ONTARIO LOAM; (*b*) AND (*e*) HEMPSTEAD SILT LOAM; AND (*c*) AND (*f*), WET SILT LOAM.

FIG. 2.—SPECIMENS OF WROUGHT-IRON PIPE, CORRODED BY EIGHT YEARS OF CONTACT WITH: (a) ONTARIO LOAM; (b) HEMPSTEAD SILT LOAM; AND (c) WET SILT LOAM.

FIG. 3.—SPECIMENS OF BESSEMER STEEL PIPE, CORRODED BY EIGHT YEARS OF CONTACT WITH: (a) ONTARIO LOAM; (b) HEMPSTEAD SILT LOAM; AND (c) WET SILT LOAM.

ences in the quantities and distribution of the impurities or strains might account for these differences in corrosion. That this was not the correct explanation was shown by the fact that, almost without exception, whenever Bessemer steel showed one form of corrosion, pure open-hearth iron, wrought iron, and specimens of Bessemer steel from another rolling mill showed similar corrosion patterns. Evidently, therefore, the corrosion pattern was primarily a function of the soil condition, and to a minimum extent only, a characteristic of the material. Although this observation was made public in 1927, and has been re-stated many times, its importance is still not fully appreciated.

Not infrequently the Bureau of Standards is asked what is the life of a certain type of pipe material. This will depend on the thickness of the pipe wall and the type of soil to which it is exposed. It will also depend on the area[3] exposed and on the definition of pipe life. Sometimes the inquirer has in mind the time until the first leak develops; others wish to know how long the pipe can be kept in service economically, which, of course, depends partly on the cost of repairs. In certain soils 8-in. steel oil lines have begun to develop leaks within six years after construction, although such early leaks are few. In other soils pipes have remained in service from 50 to more than 273 years. Only a small percentage of the soils of the United States are severely corrosive, but the individual corrosive areas are small in extent and widely distributed.

It is inadvisable therefore, to attempt to protect the entire length of a long line against corrosion unless the consequences of a few leaks would be very serious. Since underground corrosion is largely the result of certain soil characteristics many attempts have been made to correlate corrosion with soil properties and to develop methods for identifying corrosive soils. The location of corrosive soils along a pipe line is commonly known as a soil corrosivity survey or as a soil survey. The latter name is unsatisfactory because it has been previously used by the Department of Agriculture to indicate the identification and mapping of soils with respect to their physical appearance. Soil corrosivity surveys will be discussed in a subsequent section of this paper.

Soil Characteristics That Determine the Form and Rates of Corrosion.—In the course of an investigation of the relation of pipe-line currents to corrosion, Mr. E. R. Shepard[4] noted that the soils in which corrosion was most active were lower in electrical resistance than the adjacent soils. The lower resistance could be accounted for by higher percentages of water or of soluble salts. Of course, these differences would appear in complete soil analyses, but were obscured by other data.

It is not to be inferred from the fact that soils of low resistance are corrosive that the corrosivity is proportional to the conductivity of the soil or that the conductivity of the soil is in itself the cause of the corrosion. It appears, however, that the conductivity of the soil either is a factor or is

[3] "Adjustment of Soil Corrosion Pit-Depth Measurements for Size of Sample," by G. N. Scott, *Proceedings*, Am. Petroleum Inst., Vol. 14, Pt. 4, p. 204.

[4] "Pipe Line Currents and Soil Resistance, as Indicators of Local Corrosive Soil Areas," by E. R. Shepard, *Journal of Research*, Bureau of Standards, Vol. 6, p. 683, April, 1931.

associated with some soil property which influences and sometimes controls the corrosivity of the soil. Although soils of low electrical resistance are usually corrosive, exceptions are known, and some soils having high resistivity are quite corrosive. Therefore, it must be concluded that while in accordance with the electrolytic theory of corrosion the conductivity of the soil is a factor, in some instances, other factors are more important.

After a study of a wide variety of soils, Mr. I. A. Denison,[5] concluded that there was a relation between the acidity and the corrosiveness of soils. However, as in the case of the resistivity-corrosion relation, cases were found in which the correlation was unsatisfactory. Investigators are thus led to the conclusion that soil acidity is a factor affecting the rate of corrosion, but that it is not in all cases the controlling factor and, perhaps, not the fundamental one.

Consideration of underground corrosion phenomena occurring under widely different conditions leads to the conclusion that there are several, perhaps numerous, factors which modify the more fundamental causes of corrosion. In an unpublished communication to the Corrosion Committee of the American Petroleum Institute, Mr. Denison introduces his discussion of corrosion in acid soils, as follows:

"Corrosion of ferrous metals in soils may be assumed generally to originate in differences in potential between parts of the metal surface accessible to the oxygen of the atmosphere and other parts from which air is completely or partially excluded. According to this explanation, corrosion occurs at the points of contact between metal and soil which are anodic to adjacent areas more accessible to air. This general statement of the 'differential aeration' theory may be used as a basis for understanding the various forms of corrosion which are encountered in pipe line operation and to explain the marked differences which have been observed in the corrosiveness of soils."

The importance of oxygen concentration was first realized in 1916 by Mr. J. Aston,[6] although the creation of electric currents by variations in oxygen concentration was observed at a much earlier date. The importance of the phenomenon has been pointed out by Mr. W. R. Evans,[7] in a number of papers. Mr. Evans, however, has confined his attention largely to atmospheric and under-water corrosion which does not involve so many uncontrollable factors.

It now seems probable that the electrolytic theory will account for observed corrosion underground if it is assumed that the impelling differences of potential originate largely in differences of oxygen supply at various points on the surface of the buried metal, and that the potentials, or the conductance, of the path over which the corroding current flows, or both, are modified by secondary reactions such as polarization and the deposition of corrosion products.

The decrease in the rate of penetration with time manifested by the Bureau of Standards specimens and by a number of pipe systems for which

[5] "Corrosion of Ferrous Metals in Acid Soils," by I. A. Denison and R. A. Hobbs, *Journal of Research*, Bureau of Standards, Vol. 13, p. 125.

[6] *Transactions*, Am. Electrochemical Soc., Vol. 29 (1916), p. 449.

[7] "Corrosion of Metals," by W. R. Evans, Edward Arnold & Co., London, England.

records have been kept, can be explained by the protective effect of the corrosion products that are formed, or by the restriction of oxygen resulting from the settling of the soil in the trench and the increased uniformity of trench conditions. The determination of this change that is the more effective is important since, as will be shown later, it is a factor in decisions as to the effectiveness of protective coatings.

The Reduction of Corrosion Losses

To be logical and complete, this paper should contain a section dealing with the estimation of rates of corrosion and their relation to the useful life of pipe. To be helpful, such a discussion would have to be long, since the problem is a complicated one. The writer has chosen, therefore, to avoid this subject for the present and to proceed directly to a discussion of the reduction of corrosion losses. The proposed methods follow from the ideas set forth in the preceding section. The recommendations follow the form of the ancient and well-known recipe for the cooking of a hare.

Soil Corrosivity Surveys.—As one should first catch the hare, so should the engineer first locate his corrosion. The most serious corrosion, of course, will locate itself, but as soon as the engineer is surprised by the first leak in his line, he begins to wonder how much of the line is about to fail. Experience indicates that many miles of line have been needlessly uncovered and reconditioned because the owner feared that the first few leaks would be followed by others with increasing frequency. As a result of investigations at the Bureau of Standards, based on the principles of underground corrosion which have just been enumerated, several practical methods have been developed by which the pipe-line operator can determine the point of most serious corrosion. The principles and limitations of the older and more frequently used resistivity test have been described by Shepard.[4] The method has been tested in widely separated sections of the United States under several soil conditions. In all cases it has been the means of locating some of the corrosive areas. Dr. S. P. Ewing made a study of the corrosiveness of soils along the right of way of an oil line in one of the Central States, using several methods for testing the soils and comparing the results with the replacement records for the pipe line.[5] He concluded that neither soil resistivity nor soil acidity alone is a satisfactory indicator of soil corrosivity and found that by combining the two tests with information as to soil types a useful result could be obtained. He also developed a formula for determining what part of a line should be protected against corrosion and a method for determining the economic value of soil corrosion surveys.

It is scarcely to be expected that any correlation of soil conditions at a specific time with the corrosion occurring at the selected point throughout an extended period of time can be perfect, since the action of the soil is modified by the effects of temperature, oxygen, and moisture supplies, and by local conditions which often cannot be recognized. Moreover, until methods

[5] "Corrosion Surveys for Transmission Lines and Distribution Systems," *Proceedings* Am. Gas Assoc., 1934 (not yet published).

for controlling corrosive conditions are much improved, the corrosiveness of the soil cannot be expressed by a single value. When such value is available it must be understood that it is an average and that in individual cases the observed value can be expected to be greater or less by a considerable percentage. Nevertheless, values for corrosiveness are quite useful and soil surveys which indicate even in a general way that the soil in question is not corrosive, mildly corrosive, or very corrosive, may more than justify their cost.

It seems probable that if a pipe line traversing open county is to be laid or reconditioned, a systematic soil survey by a competent engineer will be justified. If the line traverses territory in which the natural soil conditions have been seriously modified by Man, as city streets containing fills of foreign material, sewer leaks, or discharges from chemical plants, or if the pipe lies beneath pavement, the economy of a survey is somewhat more doubtful, although success under these adverse conditions has been reported by Mr. A. V. Smith.[9] For the benefit of those who wish to study the subject, a list of the more important recent articles on soil corrosivity surveys is included in the Appendix.

Although probably less than 10% of the underground pipes of the country is in corrosive soil, it is likely that any extensive corrosion survey will locate one or more places where a reduction of the possible corrosive effects of the soil seems desirable. The engineer is then faced with the choice of a method for maintaining his line in one or more localities. The preferable method will depend on local conditions. Since corrosion underground has been attributed to variations in the supply of oxygen at different points along the line, an obvious method of preventing it is to secure uniform conditions with respect to oxygen. Unfortunately, the application of this principle is limited. To bury a pipe deeply is helpful unless the change in soil character and moisture with depth is unfavorable. Likewise, uniform tamping of the soil is conducive to uniform conditions. It has been suggested that the laying of a pavement over the pipe would reduce the oxygen supply and, consequently, the corrosion. This may be true only if the entire line is under pavement; otherwise, one might expect galvanic action between the pipe beneath the pavement and that in unpaved streets, with the corrosion occurring under the pavement because of the smaller quantity of oxygen there. Refilling the trench partly with less corrosive soil has been tried, but no reports as to the effectiveness of the treatment are known to the writer.

Selection of Materials.—The demand for a less corrodible pipe material is perennial, and a variety of ferrous alloys have been offered to meet this demand. The results of the investigation by the Bureau of Standards do not encourage hope in this direction if the cost of the material is considered.

It is possible that a ply-metal pipe having a corrosion-resistant metal on the outside may be developed which will not be too costly for use under conditions in which the renewal of the pipe would be expensive. Copper and high copper alloys resist nearly all soils better than ferrous materials and are

[9] "A Practical Soil Corrosion Survey with Shepard Soil Corrosivity Rods," by A. V. Smith, *Proceedings*, Am. Gas Assoc., 1932, p. 790.

being used successfully for pipes of small diameter. However, very thin copper sheet used as a shield for bituminous coatings has failed under some soil conditions.

Those demanding more resistant pipe materials may do well to consider the significance of curves representing the change in rate of penetration with time as has been previously mentioned. Whatever the cause for the decline in the rate of pitting, there can be little doubt that in many soils this decline is so great that corrosion penetrates but slowly after the first few years. In such soils the failure of pipe lines could be minimized by providing a wall sufficient to withstand the initial attack and the subsequent slow penetration. In many instances the wall thickness required for an unusually long life before penetration, does not appear to be prohibitive. It may be better engineering to provide extra wall thickness rather than to invest in corrosion-resistant materials or protective coatings.

Experience has shown that the life of many protective coatings is rather limited. It is possible, however, that the usefulness of short-lived coatings has been under-estimated. If the decrease in the rate of corrosion previously referred to is largely the result of the settling of the back-fill of the trench (that is, the establishment of more nearly uniform conditions along the pipe with respect to its contact with the soil and to the supply of moisture and oxygen), a coating which will last until these conditions have been established will eliminate that part of the corrosion-time curve which represents the period of rapid corrosion and will greatly extend the life of the pipe.

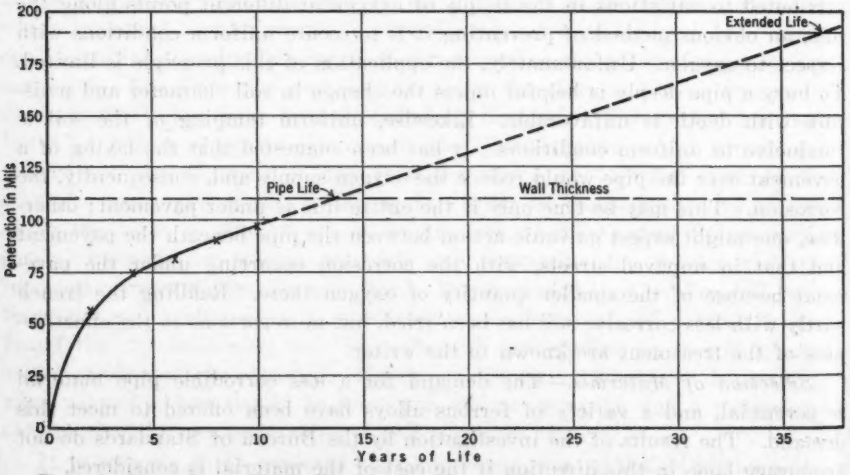

FIG. 4.—PENETRATION-TIME RELATION FOR A ¾-INCH STEEL SERVICE PIPE IN SUSQUEHANNA CLAY, AT MERIDIAN, MISS.

The effect of an increase in the thickness of the wall of a small pipe and the service performed by a temporary protective coating on such a pipe can both be seen by studying Fig. 4, the solid part of which shows the maximum penetration-time relation for wrought-iron and steel specimens in Susque-

hanna clay near Meridian, Miss. This curve is offered not as representative of corrosion in all soils, but only to illustrate the action of one soil with respect to ferrous metals. To be on the conservative side the curve has been extrapolated as a straight line, although it may bend downward because of the protective effects of corrosion products.

Consider the corrosion of a $\frac{3}{4}$-in. water service pipe the wall thickness of which is 0.11 in. It traverses a stratum of Susquehanna clay, a common Southern soil with stiff, plastic, red, heavy clay subsoil which becomes mottled in the lower part with gray, or bluish gray, red, and yellow. Fig. 4 indicates that the wall of this pipe will be punctured in a little less than 14 yr. If extra strong pipe, with a wall thickness of 0.154 in., or 40% greater, is used the life of the pipe should be approximately 26 yr, or nearly twice as long. If the standard pipe is galvanized or protected by a bituminous coating which will prevent corrosion for 5 yr (the period represented by the steep part of the curve), and if the corrosion decreases, only because of the settling of the trench as suggested previously, corrosion will begin at the end of the 5-yr period and progress as indicated by that part of the curve beyond the 5-yr point. Under this hypothesis, the application of the coating is equivalent (for corrosion-resisting purposes) to the addition of 80 mils (1 mil = 0.001 in.) to the thickness of the pipe wall, and the expected life of the pipe is extended from 14 to 37 yr, as is indicated by the intersection of the horizontal line representing 110 + 80 mils with the pitting-time curve. Thus, a coating lasting 5 yr may extend the life of a pipe 23 yr.

It must be remembered, however, that any conclusion is only as valid as the premises on which it is based. It is improbable that the decrease in the rate of corrosion is solely the result of the settling of the back-fill. Probably the formation of corrosion products plays an important part. If the settling of the trench has no effect, when the coating fails, corrosion begins as on a newly laid pipe and the extension of its life by the coating is equivalent only to the life of the coating. Probably the actual condition lies somewhere between the two extreme assumptions. The chief purpose of the present discussion is to call attention to the need for more complete and definite data on corrosion processes. Until such information is available the safe course is to provide as much protection for buried lines as can be justified.

Protective Coatings.—The idea is common that a moderately good coating is satisfactory for a moderately corrosive soil. There is no fundamental relation, of course, between the rates of deterioration of coating and of pipe since they are quite different materials and deteriorate for quite different reasons. The period of effectiveness necessary to justify the use of a coating is longer in a mildly corrosive soil than in a severely corrosive one. If a coating costs 10% of the cost of the line and if it prolongs the life of a line from 10 yr to a little more than 11 yr, it pays for itself; but if it lasts the same length of time on a line which would last 30 yr without the coating, it has increased the life of the line only 3% at a cost of 10% plus the interest on the additional investment. Recently, a water-works engineer in planning a new line, allowed a life extension of 5 yr for a coating to be applied to a

line having an estimated life, bare, of 70 yr. A little thought will show that the interest paid on the initial investment for the coating will double that investment several times in the course of the life of the line, and that the cost per year of the protected line will be considerably higher than it would have been without the coating.

In 1924, the Bureau of Standards buried a considerable number of specimens of galvanized and lead-coated pipe in a variety of soils. These coatings have deteriorated slowly and until recently it has not been possible to make statements of practical value regarding their usefulness. The Bureau's data as to the corrosion of galvanized steel indicate that the protection afforded by the zinc coating is to be ascribed to the low corrodibility of that metal rather than to the fact that the zinc is anodic to steel. Tests of 2-oz coatings of zinc applied to pipe and sheet steel indicate that rust spots may be expected in a few soils within 4 yr, whereas in other soils more corrosive with respect to steel, rust will not appear for 6 or 8 yr.

At this time a definite relation between any soil characteristic and the rate of corrosion of zinc has not been recognized. Some acid soils appear to attack the coating, whereas other soils having higher acidity do not seriously affect it. In one severely corrosive alkali soil a 2-oz zinc coating failed within 2 yr. In less corrosive alkali soils, however, the same coating has shown much more satisfactory resistance.

A comparison of the rates of penetration of lead-coated steel and plain steel does not show, definitely, an increased rate of corrosion of the steel after puncture of the lead because of electrolytic action between the two metals. Sometimes, the punctured lead coating appears to corrode more rapidly than bare steel and sometimes less rapidly. Whether the differences in the rates of corrosion are accidental or whether they are the results of different electrolytes is not known.

Research on bituminous pipe coatings has been handicapped by the inability of the Bureau to keep the same man on the problem continuously. Most of the bituminous coating work at the Bureau has been done by research associates supported by the American Gas Association and the American Petroleum Institute. The results of their work will be much more valuable after later examinations of coatings which they have had exposed to soils since 1929 and 1930. These results are most directly applicable to steel pipe lines of moderate diameters. The recommended treatment for these lines may require some modification before they are suitable for service pipes, cast-iron pipe, or pipes of large diameter.

The work indicates that the failure of properly applied bituminous coatings is usually caused by stones or earth clods. The remedy indicated is to shield or reinforce the coating by a layer of rather stiff material, such as roofing felt. For soils that are destructive to organic fabrics, a metallic or asbestos reinforcing material is preferable. As yet, the lives of the better types of bituminous coatings have not been determined. However, a sufficient number of specimens have been buried to yield satisfactory data on coating

life if the project can be continued over the period of time necessary for an adequate exposure of the coatings.

Several satisfactory methods for determining the condition of a pipe coating have been developed, but no entirely satisfactory accelerated test for coatings has been proposed. While the desirability of buying coatings on the basis of performance specifications has been recognized, no such specifications are in general use and the lack of convincing data makes the writing of such specifications, at this time, difficult and unsatisfactory.

In 1933 an electrical test[10] was developed whereby the application of a continuous coating could be insured. The writer had an oportunity to test a coating which had been applied so as to meet this continuity test. After the coated pipe had been laid in the ditch and covered, the test indicated that one or more bare spots had developed. In other words, the benefits of the careful application of the coating had been lost to a large extent because of careless treatment of the coated pipe.

The purchaser of a pipe coating should consider not only whether its fundamental properties are satisfactory but also the probability of getting it in the ground in a satisfactory condition. If the size or weight of the pipe section is so great that it cannot be handled commercially without injury to the coating, the advisability of the coating is at least debatable.

Cathodic Protection.—Fortunately, a method of treating coated pipes has been developed within the last few years which, under favorable conditions, protects the abraded parts. This is known as cathodic protection and has been well described by Mr. R. J. Kuhn[11] and by Mr. Starr Thayer.[12] Essentially, the method consists in maintaining a negative charge in the pipe line with respect to the adjacent earth by means of a superimposed current supplied by a rectifier. The economic success of the method depends on the insulating properties of the protective coating. If the coating is impervious to moisture and has not been broken in too many places, an inexpensive amount of current may prevent corrosion. Cathodic protection of bare lines, or of coated lines in metallic contact with bare lines, is not feasible because of the amount of current required.

Need for Corrosion Research

The purpose of this paper is primarily to show the status of the fight against corrosion. If the statements that have been made herein regarding rates of corrosion have been indefinite, and if the suggestions as to methods of reducing corrosion have been vague, it is because the collection of satisfactory corrosion data is necessarily slow and expensive. A definite plan of study has been outlined by students of corrosion, and definite progress has been made. The continuation of the work for a few more years should yield definite results.

[10] "The Detection of Flaws in Pipe Line Protective Coatings before Burial," by G. W. Clarvoe, *Pipe Line News*, Vol. V. No. 8, p. 13, July, 1933.

[11] "Cathodic Protection of Underground Pipe Lines from Soil Corrosion," by R. J. Kuhn, *Proceedings*, Am. Petroleum Inst., 1933.

[12] "The Development and Application of a Practical Method of Electrical Protection for Pipe Lines," by Starr Thayer, *Proceedings*, Am. Petroleum Inst., 1933.

APPENDIX

CURRENT ARTICLES ON UNDERGROUND CORROSION

Rates of Corrosion

Logan, K. H., Ewing, S. P., and Yeomans, C. D. Soil Corrosion Studies: Soils, Materials and Results of Early Observations. *Technologic Papers,* Bureau of Standards, Vol. 22, p. 447, 1928 (T 368).

Logan, K. H., and Grodsky, V. A. Soil Corrosion Studies, 1930: Ferrous Materials. *Journal of Research,* Bureau of Standards, Vol. 7, p. 1, 1931 (RP 329).

Logan, K. H. Soil Corrosion Studies: Non-Ferrous Metals and Metallic Coatings. *Journal of Research,* Bureau of Standards, Vol. 7, p. 585, 1931 (RP 359).

Logan, K. H., and Taylor, R. H. Soil Corrosion Studies, 1932. *Journal of Research,* Bureau of Standards, Vol. 12, p. 119, 1934 (RP 638).

Scott, G. N. Pit Depths as a Measure of Soil Corrosion. *Proceedings,* Am. Petroleum Inst., Vol. 14, Pt. 4, p. 204, 1933. (Adaptation of soil corrosion pit depth measurements for size of sample.)

Soil Surveys

Shepard, E. R. Pipe Line Currents and Soil Resistivity as Indicators of Local Corrosive Soil Areas. *Journal of Research,* Bureau of Standards, Vol. 6, p. 683, 1931 (RP 298).

Denison, I. A. Correlation of Certain Soil Characteristics with Pipe Line Corrosion. *Journal of Research,* Bureau of Standards, Vol. 7, p. 631, 1931 (RP 363).

Denison, I. A. Methods for Determining the Total Acidity of Soils. *Journal of Research,* Bureau of Standards, 1933, Vol. 10, p. 413, 1933. (RP 539).

Smith, W. T. How Soil Corrosiveness Can Be Measured. *Gas Age Record,* Vol. 70, p. 129, 1932.

Logan, K. H. Soil Corrosion Surveys. *Production Bulletin No. 207,* Am. Petroleum Inst., p. 142, June, 1931; also, *Oil and Gas Journal,* Vol. 30, No. 3, p. 28, June 4, 1931.

Smith, A. V. A Practical Soil Corrosion Survey. *Proceedings,* Am. Gas. Assoc., 14th Convention, p. 790, 1932.

Weidner, C. R., and Davis, L. E. Causes of Pipe Line Corrosion Scanned. *Oil and Gas Journal,* Vol. 30, No. 26, p. 41, 1931; also, *Production Bulletin No. 208,* Am. Petroleum Inst., p. 36, 1931.

Cathodic Protection

Schneider, W. R. Cathodic Protection. *Western Gas,* Vol. 9, No. 4, p. 21, April, 1933.

Thayer, Starr. The Development and Application of a Practical Method of Electrical Protection for Pipe Lines. *Proceedings,* Am. Gas. Assoc., 1933 Convention; also, *Proceedings,* Am. Petroleum Inst., Vol. 14, Pt. 4, p. 143, 1933.

Kuhn, R. J. Cathodic Protection of Underground Pipe Lines from Soil Corrosion. *Proceedings,* Am. Petroleum Inst., Vol. 14, Pt. 4, p. 153, 1933.

PROTECTIVE COATINGS

Scott, G. N. API Coating Tests Progress Reports. *Proceedings,* Am. Petroleum Inst., 1930, 1931, 1932.

Ewing, S. P. AGA Studies of Coatings for Pipe Lines. *Proceedings,* Am. Gas Assoc., p. 774, 1931.

Distribution Sub-Committee Report on Inspection and Tests of Specimens Removed in 1932. *Proceedings,* Am. Gas Assoc., p. 741, 1933.

Boyd, G. H. Pipe Coatings and Corrosion. *Proceedings,* Am. Gas Assoc., p. 737, 1933.

Clarvoe, G. W. The Detection of Flaws in Pipe Line Protective Coatings before Burial. *Pipe Line News,* 1933.

Ewing, S. P. AGA Studies of Coatings for Pipe Lines. *Proceedings,* Am. Gas Assoc., p. 774, 1931; p. 741, 1933.

Fitzgerald, Charles. Corrosion and Pipe Line Coverings. *Production Bulletin No. 20,* Am. Petroleum Inst., p. 126, 1930.

Gill, S., and Karl, F. Evaluation of Production Corrosion Losses. *Production Bulletin No. 208,* Am. Petroleum Inst., 1931.

Scott, G. N. API Coating Tests. *Production Bulletin No. 208,* Am. Petroleum Inst., p. 53, 1931.

Scott, G. N. API Coating Tests. *Production Bulletin No. 210,* Am. Petroleum Inst., p. 114.

Weidner, C. R. Protecting and Reconditioning Pipe Lines. *Bulletin No. 6,* Am. Petroleum Inst., Vol. 8, p. 352. 1927.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

REPORTS

FLOOD PROTECTION DATA[1]

PROGRESS REPORT OF THE COMMITTEE

Historical Introduction

This progress report recalls the earlier efforts by the Society to stimulate interest in the systematic collecting and listing of flood data. The movement had its inception at the time of the 1922 Spring Meeting of the Society held at Dayton, Ohio, when its program was devoted to a Symposium on Floods and Flood Control.[2] The need for research work in collecting authentic flood data for streams in all parts of the United States found strong advocates on that occasion. This led the Board of Direction in August of that year to appoint a Special Committee on Flood Protection Data composed of N. C. Grover, *Chairman*, C. B. Burdick, William P. Creager, Harrison P. Eddy, Gerard H. Matthes, Charles H. Paul, and Arthur O. Ridgway. This Committee, in the reports which it submitted during its 5-yr existence, brought out clearly the fact that in order to meet the problems of design, construction, and operation of structures situated in river channels, on river banks, and in river valleys, as adequately as is demanded of engineering design, construction, and operation of structures elsewhere, much would have to be done in the way of bringing together dependable data pertaining to past floods. It showed that although several compilations of maximum flood discharges on a world-wide basis had been published in the *Transactions* of the Society, no attempt had ever been made to compile and collate the enormous mass of available flood records pertaining to American streams in form convenient for use.

The Special Committee's principal recommendation was that for each gauging station there be listed on prescribed forms, in chronological order, all flood events above a selected basic stage, giving the date, gauge height, discharge, and a symbol expressing the degree of reliability of the record. In addition, the tabulations were to show the three greatest known floods on record, and whenever available the daily discharges during great flood waves so as to make possible the computation of total volume discharged and the platting of flood hydrographs. Printed forms[3] for these purposes

Note.—Written discussion on this Progress Report will be transmitted directly to the Chairman of the Committee for possible use in preparing subsequent reports.

[1] Presented at the Annual Meeting, New York, N. Y., January 16, 1935.

[2] *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 1383.

[3] The forms were published and described in *Transactions*, Am. Soc. C. E., Vol. 89 (1926), p. 1065.

were devised by the Special Committee together with directions for their use. For determining flood frequency, it recommended the graphic "California Method."[4]

The Special Committee held that a general flood formula or series of formulas would not serve the needs of the profession, because formulas do not make available to the user the original data on which they are based and hence do not furnish any reliable clue as regards their applicability or probable error in any given problem. It strongly advocated the publication of the original flood data so that these data might at all times be available to engineers, enabling them later to exercise to the fullest extent their own judgment in the interpretation thereof. It recommended that some agency, preferably a Federal Bureau, be created to undertake the collecting, listing, and analyzing of flood data for streams throughout the United States, this agency to maintain a repository of authentic flood data complete and reliable to a degree not attainable by either individual or co-operative effort on the part of practising engineers.

Although the Special Committee urged this program consistently in its annual progress reports, and although in June, 1925, presentations were made by the President of the Society to the Director of the Bureau of the Budget with a view to having funds made available to the Water Resources Branch of the United States Geological Survey with which to undertake this work, all efforts proved fruitless. As the magnitude of the proposed undertaking was clearly beyond the resources of the Society nothing could be done, and the Committee was dissolved by the Board of Direction during the latter part of 1928.

Subsequent events conspired to revive interest in flood data. The first of these was the extensive series of surveys and reports ordered by Congress in March, 1928, relating to the possibilities of co-ordinated development of nearly 200 rivers in the United States with respect to navigation, power, flood control, and irrigation. Known as the "308" reports, because of having originally been proposed to Congress in 1926, in House Document No. 308, 69th Congress, 1st Session, these reports dealt with floods on many streams, both in the matter of flood control and for the storage of flood waters for various proposed uses. In the preparation of these reports the Corps of Engineers, United States Army, under whose direction the majority of the "308" surveys were made, undertook compilations of flood data and flood frequency studies for many water-sheds in the United States. There was also the ever-recurring damage toll by great floods in all sections of the country. The stimulus given to water resources development by the new Administration led in 1934 to the allotment of $25 000 from funds supplied by the Federal Public Works Administration to the Mississippi Valley Committee for the compiling and analyzing of flood data and an additional $20 000 for the study of rainfall and run-off data. The Society was requested by the Mississippi Valley Committee to appoint a new Com-

[4] First used in *Bulletin 5*, Rept. to the Legislature of 1923 on the Water Resources of California, Plates LVIII to XCIII, inclusive.

mittee on Flood Protection Data to assist it in an advisory capacity in connection with the flood compilations and analyses.

The activities of the new Committee as reflected in this progress report therefore, mark the resumption of the Society's previous efforts in a field of endeavor in which it was a pioneer twelve years ago. Indications are that the Society's earlier hope of seeing the Engineering Profession provided some day with a permanent repository of authentic flood data for American streams, now bids fair to become a reality. One feature of this progress report is of especial interest in this connection; namely, the Committee's solicitude in seeing to it that these compilations and analyses of flood data shall be of maximum practical value to the profession compatible with the small sum available for doing the work.

Current Progress Report

The Committee on Flood Protection Data was appointed originally to assist the Mississippi Valley Committee of the Government in an advisory capacity in connection with compilations and analyses of flood data pertaining to Mississippi Basin streams. It began its activities in May, 1934, some time after the compilations had been started. The first meeting of the Committee was held in Washington, D. C., on August 30 and 31, soon after funds had been allotted for its use. On this occasion close contact was established with the work of compiling the data being done by personnel of the Water Resources Branch of the U. S. Geological Survey with funds supplied by the Federal Public Works Administration. Conferences were held with N. C. Grover, M. Am. Soc. C. E., Chief Hydraulic Engineer, U. S. Geological Survey, C. H. Paul and S. M. Woodward, Members, Am. Soc. C. E., of the Mississippi Valley Committee, Professor J. J. Slade, Jr., Merrill M. Bernard, M. Am. Soc. C. E., and Captain John P. Dean, Corps of Engineers, U. S. Army, engaged on special studies pertaining to flood data. As the Mississippi Valley Committee then expected to be relieved from duty on September 30, the Committee on Flood Protection Data, in submitting its recommendations included the following statement:

"This Committee feels that it is of great importance that the work of obtaining and compiling flood data be continued for the entire Mississippi Valley and be extended to cover the entire country. In its opinion carefully prepared flood data are indispensable to any intelligent consideration of problems relating to the conservation and utilization of water resources. The Committee respectfully requests that this matter be brought to the attention of the Water Resources Section of the National Resources Board, and expresses the hope that means may be found by which all flood data compiled for the Mississippi Valley Committee and the National Resources Board may be published so as to be available for the use of administrative officials throughout the United States."

About October 1, 1934, when the National Resources Board took over the work of the Mississippi Valley Committee, the compiling of flood data was extended to include the streams of the entire United States. The Committee on Flood Protection Data automatically became advisory to the Director of

the Water Resources Section of the National Resources Board. The work of compiling and analyzing flood data was continued by the Water Resources Branch of the U. S. Geological Survey. Assurance was received that the data would be made available for distribution in published form.

At its second meeting held in Washington, D. C., November 15 and 16, 1934, the Committee conferred at length with Mr. Grover, under whose general direction the work was being done, and also with R. W. Davenport and C. S. Jarvis, Members, Am. Soc. C. E., of the Water Resources Branch, who had immediate supervision over the work. As the result of this meeting recommendations were submitted by the Committee, the more important ones of which are here re-stated.

This Committee regards the compilation of flood data as of primary importance and the study of flood frequency as secondary. The greatest good to the user of flood data will result from placing before him just as complete an array of flood events as historical and engineering records can be made to yield. This relieves him of the necessity of undertaking research work of this kind at a time when his energies usually are fully occupied. On the other hand, studies to ascertain the probable frequency of recurrence of floods of various magnitudes, such as may be needed in a particular engineering problem, can readily be made by any competent engineer once he has access to adequate data. As this Committee visualizes it, the primary objective of the compilations and analyses referred to is to supply such adequate data. It understands that the original assignment requires the U. S. Geological Survey to make analyses of flood frequency and to investigate methods for making such analyses. It feels that this item has been amply covered for present purposes, but that the flood compilations could, to advantage, be perfected and extended with the remaining funds, especially as regards supplying flood-crest stages.

To make entirely clear the reasons for the Committee's position, the following considerations, based on actual practice, are submitted: In the majority of cases arising in engineering practice, the nature of the problem usually calls for some specific method of analysis, and an ordinary frequency curve based on available flood data may or may not give the required answer. Generalizations are dangerous, and all that can be done with flood data in the absence of a definite problem, is to generalize. Clearly, it is the function of the engineer in responsible charge of an important project to make such analyses as will give him the proper answer.

For instance, in bridge design, maximum recorded stages and the highest stage probable of occurrence are of prime importance, and flood frequency analyses may have to be predicated solely on river stages to the exclusion of discharges, or it may have to be based on flood discharges as well as on flood stages. In planning costly coffer-dams, the economic problem is how low a structure will answer the purpose without being overtopped by the majority of floods likely to occur during its life. The latter may be only a matter of a few months, and the problem, therefore, narrows down to a frequency study of the lesser floods probable of occurrence during those months, while great

floods will seldom be of interest. Another instance where seasonal occurrence of floods becomes of importance is in flood-control studies for the protection of agricultural lands along certain Atlantic Coast streams whose winter floods, although frequent, may be beneficial rather than detrimental to the farmer, but where summer floods, and more especially fall floods induced by tropical storms, cause intensive crop losses and, therefore, require chief consideration. Three distinct types of floods must be considered in this case, distinct not only as regards type of destructiveness but also as to meteorological origin. Separate frequency curves must be prepared, therefore, for such types of floods in problems of this kind. One general frequency curve based on all floods of record, if used in flood-control studies of the kind referred to, might lead to erroneous conclusions. Even more important is it to avoid generalization in dealing with flood data pertaining to streams subject to cloudburst run-off. The streams emerging from the eastern slope of the Rocky Mountains afford good illustrations, as cloudbursts are annual occurrences in that region. Their flood peaks normally are the result of melting snows and summer rains. If, in addition, cloudbursts occur in the watershed close enough above a gauging station so as to cause extreme concentration of run-off at that point, the resulting flood stage and discharge cannot logically be used in a general frequency analysis along with the normal flood peak data referred to. Cloudburst floods belong in a separate category and require separate consideration. In tabulations of flood data, it is desirable, therefore, to call attention to flood events which involve cloudburst run-off, as mere inspection of the tabulated figures will not bring such cases readily to light. The destructive flood of June 4, 1921, on the Arkansas River, at Pueblo, Colo., is a case of this kind.

Again, in determining the capacity of a regulating reservoir, whether for power, irrigation, flood control, public water supply, or other purposes, knowledge of the total volumes discharged by floods is of great importance, and the data should be sufficiently complete so that this information can be obtained. For detention basins for flood control, the total volume of flood discharge is likewise the controlling factor. In designing the spillway for a dam it is necessary to consider the greatest peak discharge, or the greatest 24-hr average discharge likely to occur, depending upon the conditions to be met. In the case of flood channels, levees, and non-impounding structures, peak stage and peak discharge are the important factors, while flood volume and 24-hr average discharge are of lesser significance. In designing flood-control reservoirs, it is vital to know the shortest interval of time recorded between great floods, as provision for emptying must be made accordingly. On streams along the Southern Atlantic Coast it has been observed repeatedly that two major floods have succeeded each other within ten days. This accentuates the importance of including in a flood record every occurrence above a basic stage; it also illustrates fittingly the inadequacy of any method of making frequency studies in which only the greatest flood in each year, or in each month, is considered and other floods, perhaps equally significant but not so great, are ignored.

It will be obvious from the foregoing that a Government agency such as is now engaged in compiling flood data can not possibly prepare frequency curves to cover such a variety of conditions, but it can be of considerable assistance in showing the relative values of different methods of determining flood frequency, and by publishing illustrations of various problems and methods of analysis and use of flood data. This Committee does not presume to express a preference for any particular method of frequency, or of other analysis, recognizing the fact that the nature of the problem and the quality of the available data should govern the mode of treatment to be selected. However, it is strongly of the opinion that the most desirable method in any given case will be that which will combine the following features or properties:

(*a*) It must be capable of utilizing to their fullest value all available flood data for the locality under consideration—that is, not only those contained in continuous gauge records, but also the earlier flood events concerning which definite and fairly complete data are obtainable—and should permit of giving proper weight to such data. Engineering practice is vitally concerned with the greatest floods on record, and any frequency method which fails to include all such occurrences cannot be classed as a satisfactory tool for engineers.

(*b*) Whatever the mathematical method used, it should permit of the exercise of judgment in the final adjustment of diagrammatic representations. This will be necessary nearly always when the continuous record is short, and in all cases where great historic floods are to be taken into account.

It is recognized that mathematical analyses may perform a useful function in flood studies in reducing the elements to be determined by judgment to a simple and definite form, but it is also recognized that a mathematical analysis even though technically correct can not give results more accurate than the data on which it is based, and that such analyses should not be depended upon to the exclusion of sound engineering judgment.

Whatever may be the value or convenience of statistical methods as applied to flood frequency analyses, it should be remembered, nevertheless, that data such as compose the majority of flood records rarely form a true mathematical series satisfying the requirements of the method of least squares. The irregular time intervals between flood events; the errors inherent in the observations of both stage and discharge; and changes in methods and appliances for making such observations, especially during the life of the older and longer records, conspire to destroy the homogeneity and comparability of the flood events viewed as a strict mathematical series, or as a sample truly representative of a long period of time. This emphasizes the value of graphical methods of analysis in which data of unequal weights may be earmarked and assigned such weights as engineering judgment may dictate.

This Committee recognizes that there is a natural maximum limit to the flood-producing capabilities of a given drainage basin, and feels that study

directed to the determination of these limiting flood-discharge rates, either by statistical means or from meteorological considerations, or both, is of more importance than the determination of frequencies of occurrence of floods of lesser volumes.

Tabulations showing comparisons of flood flows expressed in terms of calendar-day averages, maximum 24-hr averages, and peak discharges appear to the Committee as of considerable value as they invite attention to the relative differences between average and peak discharges from different kinds of water-sheds. These differences are greater for flashy streams than for slowly rising and falling streams, and, in general, are greater for streams draining small water-sheds than for large rivers, which emphasizes the desirability of extending such studies to include data for drainage areas ranging in size from less than 100 sq miles, by convenient increments, up to 10 000 sq miles and greater.

The Committee urges that the National Resources Board make recommendation that suitable provision be made: First, for continuing the work of compiling flood data to cover all streams on which gauges are maintained, to the end that a repository for such data be created and maintained permanently, provision to be made for extending the records as time goes on by the addition of new as well as historic data as they become available; second, for the co-ordination of all stream-gauging operations and maintenance of gauge records now being handled by divers Bureaus of the Government by one Federal agency which will have charge of the analyses and publication of stream and other hydrologic data, including flood data; third, that some form of supervisory agency be set up to direct and advise such work, this latter agency to include members of the Engineering Profession not engaged in Government work, as well as members of Government organizations having occasion to use such data but not engaged in their collection. One important function of such a supervisory agency would be to prevent lapses in the collection of essential data and to see that there is adequate unification of methods and correlation of the data with regard to the different uses which may be made thereof, and to prevent duplication of effort. This should increase greatly the scope and efficiency of the data and at the same time should reduce materially the cost of collection. For example, it should be possible to maintain in conjunction with an important gauging station on a stream, an evaporation record, a river-stage record, and a rainfall intensity record.

This Committee senses that more effective water utilization, such as will result from the proposed National program of development of the water resources of the United States, necessarily will affect natural stream flow; it will cause material modification of flood stages and discharges, and will make it increasingly difficult in the future to secure records representing natural stream flow. This condition has already made itself felt on a number of streams. In view of this impending situation, it appears especially important that stream-flow operations and the compiling of flood records be pushed vigorously at this time in the history of the United States. The success of

the future utilization of the water resources of the United States, in all its phases, is dependent upon the adequacy and accuracy of the records of river stages and stream flow data.

Respectfully submitted,

GERARD H. MATTHES, *Chairman*,
FREDERICK H. FOWLER,
ROBERT E. HORTON,
IVAN E. HOUK,
CHARLES W. KUTZ,
CHARLES W. SHERMAN,
DANIEL C. WALSER,

December 26, 1934. Committee on Flood Control Data.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
REPORTS

MASONRY AND REINFORCED CONCRETE[1]

PROGRESS REPORT OF COMMITTEE OF THE STRUCTURAL DIVISION

To the Executive Committee,
Structural Division,
American Society of Civil Engineers:

The Committee of the Structural Division on Masonry and Reinforced Concrete is divided into six Sub-Committees, for the study of design problems in reinforced concrete in connection with: (*A*) Rigid corners; (*B*) frames under wind loads; (*C*) volume changes; (*D*) columns; (*E*) two-way slabs; and (*F*) irregular flat slabs.

Sub-Committee (*A*) on the Design of Rigid Corners in Reinforced Concrete

Late in 1933 it came to the attention of Sub-Committee (*A*) that an elaborate study of this problem is under way in a Mid-West university. Accordingly, this Sub-Committee is marking time until the investigation is completed and has been reported.

Sub-Committee (*B*) on Reinforced Concrete Frames Under Wind Load

Although it presents no report at this time, Sub-Committee (*B*) is actively at work.

Sub-Committee (*C*) on the Influence upon Structural Design of Volume Changes and Plastic Flow in Concrete

The objects of Sub-Committee (*C*) are: (1) To indicate the nature of stress distribution in reinforced concrete members under conditions of sustained stress and variations in moisture content; and (2) to recommend modifications in design procedure and design specifications to take into account the effect of volume change.

Three viewpoints are being considered: (1) That of the scientific investigator, who is interested in observing the effect of individual variables upon the magnitude of volume change; (2) that of the structural designer, who

Note.—Written discussion on this Progress Report will be transmitted directly to the Chairman of the Committee for possible use in preparing subsequent reports.

[1] Presented at the meeting of the Structural Division, New York, N. Y., January 17, 1935.

would like to include in his design formulas some simple coefficient to allow for the effects of volume change and who would like more adequate criteria for such matters as joint spacing; and (3) that of the constructor, who would like some working rule with regard to job procedures which will reduce volume change to a minimum.

So far, the work of the Sub-Committee has been confined to studies of existing data and to the making of preliminary tests on concrete to define the factors which must be considered in approaching the structural aspects of the problem. The Sub-Committee is unanimous in the opinion that relatively little of the existing information (except with regard to reinforced columns) can be utilized directly by the designer; but from the studies made it has been possible to formulate the general nature of necessary future investigations.

Nature and Significance of Existing Data.—The available information may be evaluated upon two bases: (1) With respect to theories of the fundamental nature of the phenomena; and (2) with respect to an indication of the effect and extent or limit of flow and shrinkage.

Some advance has been made toward an understanding of the nature of flow and shrinkage. It is evident that flow and shrinkage are related, and although their investigation may require different techniques, in their relation to structures they should be considered simultaneously.

The problem of crack formation with relation to volume changes is one of extreme importance, and it is believed that the investigation cannot hope to approach a satisfactory solution until this phenomenon is understood more thoroughly. Crack formation enters as the most seriously disturbing element in any consideration of the variation of stress across the section of structural members.

In general, it is known that the practical effect of plastic flow is to redistribute and relieve the stress in the concrete of reinforced members, and to increase the stress in the steel.

Investigations of Plastic Flow.—Extensive tests on columns have been conducted by the American Concrete Institute,[2] which have yielded valuable information. Recent tests[3] on rigid concrete frames have indicated that the distribution of moments is not greatly affected by plastic flow.

Investigations of Hygral Deformations.—The literature on hygral deformations of concrete is abundant, but there is little correlation between the results of various investigators. Exploratory tests by members of Sub-Committee (*C*) have demonstrated that concrete does not shrink uniformly in all directions, but that shrinkage occurs largely at right angles to the direction of moisture travel. Since in several service structures shrinkage measurements were attempted only at the surface and in the direction of moisture travel (because in that direction restraint was a minimum), the indicated results did not represent the true shrinkage in the direction which affected the performance of the structure.

[2] "Reinforced Concrete Column Investigation," *Journal*, Am. Concrete Inst., February, 1933.

[3] "Plastic Flow in Rigid Frames," by Richart-Brown-Taylor, *Journal*, Am. Concrete Inst., January–February, 1934.

Structural Aspects of Flow and Shrinkage.—Following, in outline form, is a preliminary classification with regard to various types of structures in terms of: (1) The probable relative importance of the several kinds of volume change; and (2), the suggested treatment in design. Correspondents are invited to consider and to suggest modifications of this outline, which is considered important inasmuch as it may serve as a guide in future studies.

Building Frames.—Members are relatively long and narrow. In this case, strains and thermal deformations may be handled separately from other types of volume change and according to present methods. The following detailed discussion of the various members includes columns, beams or girders, monolithic frames, and walls.

(*a*).—Columns.—The average effect of flow and shrinkage in columns is to increase steel stress with the passing of time. Since the steel is placed uniformly and on the outside of a column, the problem of columns under axial load might be solved on the basis of average deformation of the member; that is, variations across the section need not be taken into account. Lateral dimensions affect this problem.

(*b*).—Beams or Girders.—Variation in deformations across the section of a beam or girder must be considered, although in many cases the hygral deformations may be considered as uniform across the section. Factors affected are: (1) Deflection or warping of the beam; (2) change in steel stress; and, (3) change in length of the beam with resulting axial tension if the abutments are rigid, resulting in cracking across the entire beam.

(*c*).—Monolithic Frames.—Flow and shrinkage probably have little effect upon moments in the frame considered as a whole.

(*d*).—Floor-Slabs.—Consideration should be given to sagging, surface cracking, and cracking due to tension set up by floor-slabs shrinking away from abutments.

(*e*).—Walls.—Contraction during the process of drying is of greatest importance in the case of walls; flow is relatively unimportant. Consideration should be given to surface cracking, and to vertical or diagonal cracks due to horizontal shrinkage, with horizontal restraint that varies from almost full restraint at the foundation to almost no restraint at the upper stories.

Semi-Massive Structures, Class A.—In the category of semi-massive structures are included footings, piers, small abutments, retaining walls, roadway slabs on subgrade, and bridge decks. Elastic and thermal deformations may be treated separately by current methods, and offer no problem in the present case. Flow is not of direct importance; but hygral deformations are of principal importance. In the problem of stresses due to hygral deformations, flow tends to relieve the stresses and to delay cracking. Thus, the hygral-stress problem involves an estimate of the extensibility of the concrete as well as the tendency to contract upon drying.

In cases of no end restraint (as for a pier or for short abutments) the variation in hygral deformation across the section results in surface cracking and, therefore, in decreased resistance to weathering. In road-slabs and

certain types of footings, the difference in exposure of the two sides makes the variation of hygral deformations across the section important; surface cracking and "curling" are also involved. In long retaining walls and abutments, road-slabs, and certain footings, the average shrinkage is important in determining the frequency of through cracks and the spacing of contraction joints.

Semi-Massive Structures, Class B.—Arch ribs and heavy girder bridges are included in the catagory of semi-massive structures, Class *B*. In these structures elastic deformations due to live load and thermal deformations due to rapid changes of atmospheric temperature may be handled separately by present methods of analysis. In arches, plastic deformations are important and should probably be studied separately on the basis of average deformation across the section. Shrinkage should be considered with respect to both average effect and variation across the section. In heavy girders, the problem of sagging and change in steel stress should probably be dealt with by consideration of variation in flow and shrinkage, acting simultaneously, across a section.

In both types of structures, stresses due to a rise in temperature resulting from hydration of the cement may be a factor and may require a special analysis for each case. Thermal deformations and plastic deformations, as well as the other thermal and elastic properties of the material, are involved.

Massive Structures; Dams and Large Abutments.—Elastic stresses due to short-time loads, and foundation stresses due to gravity loads may be calculated by the usual methods. In thick dams the effect of changes in atmospheric temperature may be of small importance and in any event would be studied in combination with thermal changes throughout the dam due to heat of setting and its dissipation. In thin arch dams, the effects of changes at atmospheric temperature would probably be best handled separately. In arch dams, the problem of plastic, combined with thermal, deformations may be of great importance; and the nature of the stress action requires consideration of changes in three dimensions. Except in so far as surface cracking is concerned, hygral deformations may be of little importance in dams.

Present Plans of Sub-Committee (C).—The work which the Sub-Committee has outlined for itself for the immediate future is, as follows:

(1) An attempt to correlate existing information on free hygral deformation of small laboratory specimens;

(2) Continued study and laboratory investigation of the fundamental nature of plastic deformation;

(3) Continued laboratory investigation and study of the "directional" effect in hygral deformations; that is, shrinkage along lines normal and parallel to the direction of moisture loss and at various distances from the surface from which the loss takes place. (This will include preliminary laboratory investigations of the effect of size of mass upon hygral deformations); and,

(4) Assuming various types of flow and shrinkage, to make an analytical study of stress distribution in structural members due to these causes.

Work which it is desirable to inaugurate in the near future, but which, at present, is impossible because of lack of funds, is as follows:

(*a*) Comprehensive investigations of the shrinkage throughout the mass in masses of various sizes and shapes and for various exposures;

(*b*) Laboratory investigations of the effect of flow and shrinkage upon stresses and deflections in beams having restrained ends, under controlled conditions of drying;

(*c*) Further laboratory investigations of the effect of flow and shrinkage on moments and stresses in rigid frames; and

(*d*) A comprehensive series of investigations to determine the relation between volume change of concrete exposed to job conditions and concrete specimens subjected to a "standard exposure cycle" in the laboratory.

Sub-Committee (*D*) on Reinforced Concrete Columns

Sub-Committee (*D*) was organized in the summer of 1934. It proposes to co-ordinate existing laboratory data on columns axially loaded and to study certain phases of the application of these tests to design procedure.

It is hoped that the work of the Sub-Committee may define more clearly certain questions relating to the effect of secondary flexure in concrete columns. Correlation of analytical and experimental viewpoints with observations on existing structures, in connection with the effects of continuity between beams and columns, is essential for further progress in the study of continuity and for proper selection of working stresses in columns. A series of tests on concrete columns eccentrically loaded is now in progress at the University of Illinois, and the Sub-Committee plans to keep in touch with this investigation.

Sub-Committee (*E*) on Reinforced Concrete Slabs Supported on Four Sides

The unique provisions relating to two-way slabs of the pending building code for New York City have directed attention to the inadequate nature of present design knowledge for this important form of floor construction. Sub-Committee (*E*) was organized in December, 1934, and has not had time to do more than make a beginning at its task.

Sub-Committee (*F*) on Moment Coefficients for Irregular Types of Flat Slabs

In spite of considerable progress in applying the results of analytical work to the design of flat slabs, the available data have not yet been exhausted. Current design coefficients for regular flat slabs are based upon the analysis published in 1921 by H. M. Westergaard, M. Am. Soc. C. E.[4] Additional data in his paper are available for determination of moment coefficients

[4] "Moments and Stresses in Slabs," by H. M. Westergaard, *Proceedings*, Am. Concrete Inst., 1921.

for some irregular types of flat slabs. From this study Sub-Committee (*F*) has prepared two complete sets of design specifications as follows:

(1) Regulations for the design of flat slabs arranged in two rows of panels, and arranged in four adjacent corner panels, using the two-way type of reinforcement; and

(2) Regulations for the design of flat slabs without column capitals, or with capitals limited to $\frac{c}{l} = 0.125$, using the two-way type of reinforcement.

The final report of Sub-Committee (*F*) has been completed, but has not yet been submitted to the entire Committee for study and action. It is expected to be ready for early publication.

PERSONNEL

The personnel of the various Sub-Committees is, as follows:

Sub-Committee (*A*).—C. B. McCullough, *Chairman,* Thor Germundsson.

Sub-Committee (*B*).—C. T. Morris, *Chairman,* D. C. Coyle, Albert Smith.

Sub-Committee (*C*).—R. E. Davis, *Chairman,* R. W. Carlson, H. E. Davis.

Sub-Committee (*D*).—Hardy Cross, *Chairman,* L. S. Moisseiff, C. S. Whitney.

Sub-Committee (*E*).—Ben Moreell, *Chairman,* J. R. Nichols.

Sub-Committee (*F*).—Thor Germundsson, *Chairman,* L. H. Corning.

Contact Member.—J. F. Leonard.

Representative of American Railway Engineering Association.— F. R. McMillan.

Representative of American Society for Testing Materials.—H. H. Quimby.

For the Committee of the Structural Division on Masonry and Reinforced Concrete,
HALE SUTHERLAND,
Chairman.

January 16, 1935.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ANALYSIS OF SHEET-PILE BULKHEADS

Discussion

BY DR. ING. E. LOHMEYER

DR. ING. E. LOHMEYER[33] (by letter).[33a]—The calculations pertaining to the tests described in this paper are based on the method developed by Dr. Ing. Blum,[34] and first published by the writer[35] in 1930. The tests are very instructive, and Mr. Baumann's extremely thorough and expert mathematical study indicates particularly the direction for further scientific treatment of the extraordinarily complicated problems of the passive resistance of soil. It is to be regretted that additional tests with varied loads, could not have been run. Nevertheless the data presented confirm the usefulness of Blum's method, which the writer recommended for the calculation of bulkheads several years ago, for the reason that, since it takes account of restraint at the bottom, it approaches essentially closer to actual conditions than any other method of calculation used to that time. Mr. Baumann's experiments result in a somewhat different distribution of the passive resistance, the most important results, however (the position of the "substituted" support and the depth of penetration) remain substantially unaltered. Henceforth, it will not be necessary to base such calculations on curved surfaces of rupture, as developed by Mr. Baumann, because present knowledge of the action of soil is so incomplete that the more precise, but more elaborate, calculation is not worth the trouble.

The writer endorsed the Blum method as a particularly short rapid one, which makes it possible to gain a quick understanding of the influence of the fundamental premises concerning soil properties, the internal friction of the soil, water levels, etc. Changes in these premises within the possible

NOTE.—The paper by Paul Baumann, M. Am. Soc. C. E., was published in March, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: In May, 1934, by Jacob Feld, M. Am. Soc. C. E.; August, 1934, by Messrs. R. L. Vaughn, M. A. Drucker, and Raymond P. Pennoyer; October, 1934, by D. P. Krynine, M. Am. Soc. C. E.; November, 1934, by Dr. Ing. E. h. O. Franzius; and December, 1934, by Theodor von Kármán, M. Am. Soc. C. E.

[33] Oberbaudirektor i. R., Berlin, Germany.

[33a] Received by the Secretary December 24, 1934.

[34] "Einspannungsverhaeltnisse bei Bohlwerken", von Dr. Ing. Blum, Berlin, 1931, Wilhelm Ernst & Sohn.

[35] "Der Grundbau", von Brennecke-Lohmeyer, 4. Aufl., pp. 61–92, Berlin, 1930, Wilhelm Ernst & Sohn.

(and also within the probable) allowable limits nearly always alter the results of the calculation to a greater degree than the method of calculation no matter how carefully that may be applied. Nevertheless, the scientific value of the more precise calculation for the development of earth-pressure investigations as proposed by the author remains unchallenged. In special cases of unusually large bulkheads it will be used without question, particularly when the action of the soil is known beyond doubt, which means when the soil lacks cohesion.

In any case, one cannot be too conservative in assigning a numerical value to the passive resistance of a toe slope under water in front of the bulkhead; if there is any doubt one should consider that resistance as zero. However, the writer does not wish to enlarge on this question; he would rather confine himself to one phase—the problem of the section modulus of the interlocking steel sheet-pile discussed in the paper.

When drawing his conclusions from the deflections measured during the tests in regard to the section modulus of the sheet-piling, Mr. Baumann made a mistake, which the writer himself has also made in the past.[36] In Table 2 the proportion of the moments of inertia, I, is exactly equal to the section modulus. Correspondingly, in Table 3, the section modulus, S_e, has been computed from the moments of inertia, I_e, according to the equation:

$$S_e = \frac{I_e}{e} \qquad (105)$$

by entering e (a constant) as the distance of the extreme fiber from the axis of symmetry of the section.

The concept of the moment of resistance of a beam originates with the general equation:

$$f = \frac{M\,e}{I} = \frac{M}{S} \qquad (106)$$

in which, f = the stress in the extreme fiber of the cross-section; M, the bending moment resisted by the beam; I, the moment of inertia of the cross-section of the beam in relation to the gravity axis; and, e, the distance of the extreme fiber from the neutral axis.

In the case of simple bending the neutral axis coincides with the gravity axis and e is constant. If the beam is subjected to an axial force in its longitudinal direction, however, the neutral axis and the gravity axis no longer coincide, and e varies in proportion to the bending moment and the longitudinal force.

Consider a deep, arched, sheet-pile wall with interlocks, in the axis of the wall, set so tightly that they may be considered as a unit. In this case, the piles are subjected to simple bending only, the wall bends around the gravity axis, n-n, the neutral axis coincides with the gravity axis, and the stress is as indicated in Fig. 24(a). If there is no friction or "jamming" in the interlock to oppose the reciprocal shifting of the piles—or, in other words, if there is no shearing force transmitted in the interlock—so that piles can slide in it

[36] *Bautechnik*, 1927, pp. 26, 49, and 73.

without resistance, then the individual piles are likewise subjected to simple bending. In that case, however, they deflect around their own axes of gravity, $x - x$ (Fig. 24), the neutral axis for each pile lies in this axis of gravity, and

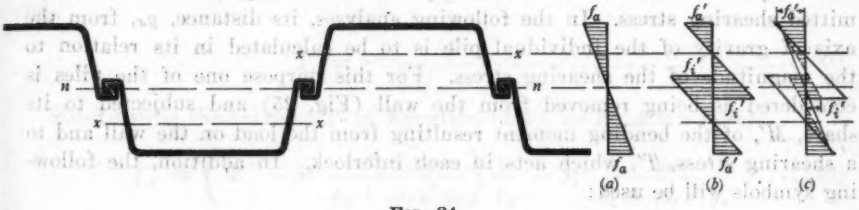

FIG. 24

the resulting stress diagram is as shown in Fig. 24(*b*). For the calculation of the stress, I and e must be entered in Equation (106) in relation to the axis, $x - x$.

The distribution of stress varies between the two boundary cases illustrated by Fig. 24(*a*) and Fig. 24(*b*), respectively. It depends on the amount of the shearing force which the interlocks of one pile can transmit to the other. As long as the shearing force created in the interlock by the load cannot be entirely absorbed, one cannot base the calculation of the extreme fiber stress upon the wall as a unit. The designer must begin with the individual pile that is subjected to its respective share of the bending moment resulting from the total load, and to two longitudinal forces that are caused by the transmission of the shearing force of the neighboring pile in the interlock. If this shearing force equals zero, the individual piles are subjected to a pure bending stress, and the neutral axis lies in the axis of gravity of the individual pile, $x - x$. If the shearing stress increases, the compression or tension stress from the shearing force is added to the pure bending stress, and the neutral axis moves from the position, $x - x$, parallel to the wall axis, $n - n$ (as demonstrated in Fig. 24(*c*)) which it reaches, if the interlock is capable of taking care of the full shearing stress. The extreme fiber stresses are to be calculated according to Equation (106) by inserting I in relation to the axis of gravity of the individual piles, and e as the distance of the extreme

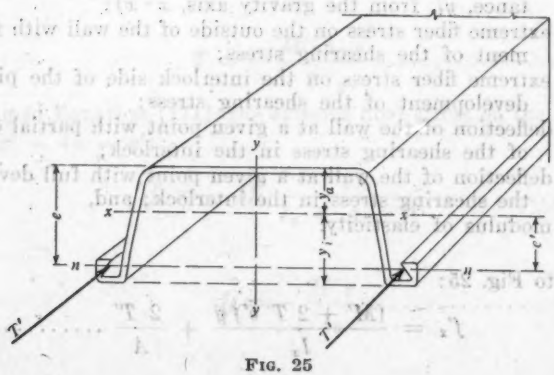

FIG. 25

fiber from the neutral axis. This means that e is variable if the shearing stress is between zero and the full value, and in that case it cannot be assumed as a constant, as Mr. Baumann has done.

The position of the neutral axis depends on the magnitude of the transmitted shearing stress. In the following analyses, its distance, y_x, from the axis of gravity of the individual pile is to be calculated in its relation to the magnitude of the shearing stress. For this purpose one of the piles is considered as being removed from the wall (Fig. 25) and subjected to its share, M', of the bending moment resulting from the load on the wall and to a shearing stress, T', which acts in each interlock. In addition, the following symbols will be used:

I_x = moment of inertia of the individual pile in relation to its axis of gravity, x-x;
I_n = moment of inertia of the individual pile in relation to the wall axis, n-n;
A_f = cross-sectional area of the individual pile;
e = distance of the extreme fiber from the wall axis, n-n;
e' = distance of the axis of gravity, x-x, of the individual pile from the wall axis, n-n;
y = distance of a random point in the cross-section, A_f, from the axis of gravity, x-x, of the individual pile (positive downward);
y_x = distance of the neutral axis from the axis of gravity, x-x, of the individual pile;
y_a = distance of the extreme fiber opposite the interlock side of the individual pile, from the axis of gravity, x-x;
y_i = distance of the extreme fiber on the interlock side of the individual pile from the axis of gravity, x-x;
T' = the shearing force transmitted to each interlock of the examined pile from the neighboring pile;
T = the maximum value of the shearing force in the interlock that develops if the interlock is capable of absorbing the entire shearing force caused by the load;
f'_x = tension at any given point in the cross-section, A_f;
f'_a = extreme fiber stress on the outside of the individual pile (at a distance, $y_a = -[e - e']$, from the gravity axis, x-x);
f'_i = extreme fiber stress of the interlock side of the pile (at a distance, y_i, from the gravity axis, x-x);
f_a = extreme fiber stress on the outside of the wall with full development of the shearing stress;
f_i = extreme fiber stress on the interlock side of the pile with full development of the shearing stress;
δ' = deflection of the wall at a given point with partial development of the shearing stress in the interlock;
δ = deflection of the wall at a given point with full development of the shearing stress in the interlock; and,
E = modulus of elasticity.

According to Fig. 25:

$$f'_x = \frac{(M' + 2\,T'\,e')\,y}{I_x} + \frac{2\,T'}{A} \quad \ldots\ldots\ldots\ldots (107)$$

The equation for the neutral axis is obtained by substituting $f_x' = 0$ in Equation (107); thus:

$$y_x = - \frac{2\,T'\,I_x}{A_f\,(M' + 2\,T'\,e')} \qquad \ldots\ldots(108)$$

and,

$$T' = - \frac{M'\,A_f\,y_x}{2\,(I_x + A_f\,e'\,y_x)} \qquad \ldots\ldots(109)$$

Substituting the value of T' in Equation (109), in Equation (107):

$$f'_x = \left(M' - M' \frac{A_f\,e'\,y_x}{I_x + A_f\,e'\,y_x} \right) \frac{y}{I_x} - M' \frac{y_x}{I_x + A_f\,e'\,y_x} \qquad \ldots\ldots(110)$$

or,

$$f'_x = \frac{y - y_x}{I_x + A_f\,e'\,y_x}\,M' \qquad \ldots\ldots(111)$$

For the extreme case (see Fig. 24(*b*)) in which no shearing force is transmitted in the interlock (that is, for $T' = 0$), the neutral axis coincides with the gravity axis, x-x, of the individual pile, and $y_x = 0$. The extreme fiber stresses of the pile in this case are expressed by Equations (107) and (110), thus: On the outside of the pile, with $y = -y_a = -(e - e')$:

$$f'_a = - \frac{M'\,(e - e')}{I_x} \qquad \ldots\ldots(112)$$

and on the interlock side of the pile, with $y = y_i$:

$$f_i = + \frac{M'\,y_i}{I_x} \qquad \ldots\ldots(113)$$

For the extreme case (see Fig. 24(*a*)), in which the shearing force resulting from the load is fully taken up in the interlock (that is, for $T' = T$), the neutral axis coincides with the axis of the wall, n-n; and $y_x = e'$. Then, the shearing force according to Equation (109) is,

$$T = - \frac{M'\,A_f\,e'}{2\,(I_x + A_f\,(e')^2)} \qquad \ldots\ldots(114)$$

and,

$$T = - \frac{M'\,A_f\,e'}{2\,I_n} \qquad \ldots\ldots(115)$$

and according to Equation (111), the extreme fiber stress on the outside of the pile, with $y = -(e - e')$:

$$f_a = - M' \frac{e}{I_x + A_f\,(e')^2} \qquad \ldots\ldots(116)$$

or,

$$f_a = - \frac{M'\,e}{I_n} \qquad \ldots\ldots(117)$$

The value for f_a in Equation (117) also results directly from Equation (106). The magnitude of the extreme fiber stress, f_i, on the interlock side of the pile is insignificant.

The deflection at a given point follows the law:

$$\delta_x = \frac{1}{E\,I}\int M\,x\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (118)$$

in which, the quantity, $\int M\,x\,dx$, represents the statical moment of the moment area in relation to the vertical through the support. The moment area of the individual pile consists of the pile's share, M', of the bending moment from the load and of the moment resulting from the transmitted shearing force, T'. If the value, T', from Equation (109) is inserted:

$$\delta' = \frac{1}{E\,I_x}\int (M' + 2\,T'\,e')\,x\,dx$$

$$= \frac{1}{E\,I_x}\int M'\left(1 - \frac{A_f\,e'\,y_x}{I_x + A_f\,e'\,y_x}\right)x\,dx \quad \ldots\ldots\ldots\ldots (119)$$

or,

$$\delta' = \frac{1}{E}\int \frac{M'\,x}{I_x + A_f\,e'\,y_x}\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (120)$$

One may assume with sufficient approximation that, for equal loads, the share of the shearing force absorption, and consequently, the distance, y_x, from the gravity axis, x-x, is equal within the boundaries of the moment area, so that:

$$\delta' = \frac{1}{E\,(I_x + A_f\,e'\,y_x)}\int M'\,x\,dx \quad \ldots\ldots\ldots\ldots (121)$$

If the shearing force is fully absorbed in the interlock, $y_x = e'$, and, hence, the deflection,

$$\delta = \frac{1}{E\,(I_x + A_f\,(e')^2)}\int M'\,x\,dx = \frac{1}{E\,I_n}\int M'\,x\,dx \quad \ldots\ldots (122)$$

The value for δ in Equation (122) also results directly from Equation (118).

With the assistance of the foregoing equations, one can determine for each value of the shearing force acting in the interlock of the piles the relevant extreme fiber stresses and deflections in proportion to the acting extreme fiber stresses, and deflections in case of complete absorption of the shearing force.

For this purpose one introduces the following proportions, which result for the shearing force from Equations (109) and (114), for the extreme fiber stresses from Equations (110) and (116), and for the deflections from Equations (121) and (122); thus:

$$\frac{T'}{T} = \frac{y_x}{e'} \times \frac{I_x + A_f\,(e')^2}{I_x + A_f\,e'\,y_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (123)$$

$$\frac{f'_x}{f_a} = -\frac{y - y_x}{e} \times \frac{I_x + A_f\,(e')^2}{I_x + A_f\,e'\,y_x} \quad \ldots\ldots\ldots\ldots\ldots (124)$$

and,

$$\frac{\delta'}{\delta} = \frac{I_x + A_f (e')^2}{I_x + A_f e' y_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (125)$$

For instance, for a certain sheet-pile section, the following standard characteristics are published: $I_x = 2\,097$ cm^4; $A_f = 58.6$ cm^2; $e = 13.5$ cm; $e' = 7.34$ cm; $y_i = 9.49$ cm; and, $y_a = -6.16$ cm.

Substituting these values, the expression common to Equations (123) to (125) will be:

$$\frac{I_x + A_f (e')^2}{I_x + A_f e' y_x} = \frac{2\,097 + 58.6 \times 7.34^2}{2\,097 + 58.6 \times 7.34 \times y_x}$$

$$= \frac{5\,253}{2\,097 + 430.1\, y_x} = \frac{12.213}{4.876 + y_x}$$

and, according to Equations (123) to (125):

$$\frac{T'}{T} = \frac{12.213 \times y_x}{7.34\,(4.876 + y_x)} = \frac{1.664\, y_x}{4.876 + y_x} \quad \ldots\ldots\ldots\ldots (126)$$

$$\frac{f'_a}{f_a} = \frac{6.16 + y_x}{13.5} \times \frac{12.213}{4.876 + y_x} = \frac{0.905\,(6.16 + y_x)}{4.876 + y_x} \quad \ldots\ldots (127)$$

$$\frac{f'_i}{f_a} = \frac{0.905\,(9.49 - y_x)}{4.876 + y_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (128)$$

and,

$$\frac{\delta'}{\delta} = \frac{12.213}{4.876 + y_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (129)$$

These values were calculated for different distances, y_x, of the neutral axis from the axis of gravity of the individual pile, and the results plotted in Fig. 26(*a*), beginning with the position of the neutral axis; that is, for the different magnitudes of y_x. For each value of y_x one can easily begin with the share of the shearing force corresponding to this position (which is absorbed by the interlock) and the increase of the extreme fiber stresses and deflections as compared to the condition where the shearing force is fully transmitted in the interlock. This process has been repeated in Fig. 26(*b*), except that the values of y_x have been so distorted that the plotted values of $\frac{T'}{T}$ appear as a straight line. Consequently, the curves for the stresses and deflections also become straight and the stresses and the deflections increase uniformly with the decrease of the shearing force transmitted in the interlock.

If the interlock cannot absorb the shearing force fully, the extreme fiber stresses and the deflections increase with the diminishing transmission of shear. Compared to the maximum stress, which appears with the shear transmission in the interlock, the extreme fiber stress, f'_a, of the outer fiber grows slowly with an equal load, but with decreasing shear transmission. The

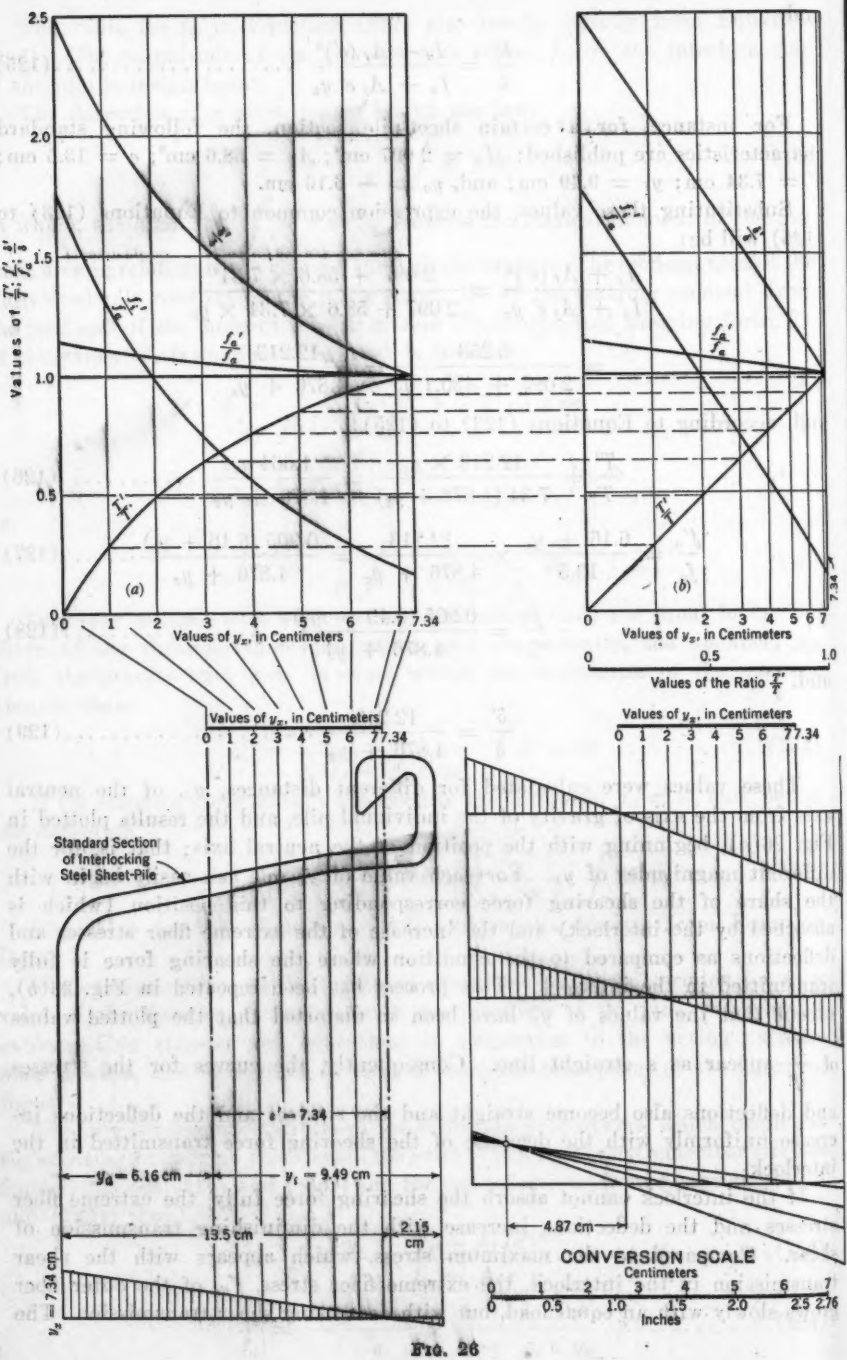
Values of $\frac{T'}{T}; \frac{f_a'}{f_a}; \frac{y'}{y}$
(a)
(b)
Values of y_x, in Centimeters
Values of the Ratio $\frac{T'}{T}$
Standard Section of Interlocking Steel Sheet-Pile
$y_a = 6.16$ cm
$y_s = 9.49$ cm
$s' = 7.34$
13.5 cm
2.15 cm
$y_z = 7.34$ cm
4.87 cm
CONVERSION SCALE
Centimeters
Inches

FIG. 26

extreme fiber stress, f'_i, of the inner extreme fiber of the pile (on the interlock side) grows faster from its originally insignificant value (0.16 f_a), and reaches, with $f'_i = f'_a = 1.08\ f_a$, the value of the outer extreme fiber stress, if the interlock absorbs only 0.42 of the full shearing force. Hence, $f'_i > f'_a$.

A summary of the values corresponding to the two boundary cases, and the conditions under which $T' = 0.42\ T$, and $f'_i = f'_a$, is given in Table 6.

TABLE 6.

Shearing force transmitted in interlock (1)	Deflection (2)	Outer extreme fiber stress (3)	Inner extreme fiber stress (interlock side) (4)	Equivalent moment of resistance (5)
$= T'$	$= \delta'$	$= f'_a$	$= f'_i$	$= S'_e$
$= 1.0\ T$	$= 1.00\ \delta$	$= f_a$	$= 0.16\ f_a$	$= S_e$
$= 0.42\ T$	$= 1.87\ \delta$	$= 1.08\ f_a$	$= 1.08\ f_a$	$= 0.926\ S_e$
$= 0$	$= 2.51\ \delta$	$= 1.15\ f_a$	.....	$= 0.87\ S_e$
		.....	$= 1.765\ f_a$	$= 0.567\ S_e$

In Column (5), Table 6, the "equivalent" section modulus, S'_e, is computed as proportional to the full "catalog value," S_e, relative to the axis of the wall. Introducing this section modulus, S'_e, in Equation (106), one arrives at the extreme fiber stress resulting from the load.

The foregoing values are valid for a particular type and size of standard sheet-pile. For other sizes of the same type, similar values apply, if anything, less favorably than the foregoing.

The aforementioned calculation permits a definite conclusion that the extreme fiber stress in the standard sheet-piling under discussion increases only insignificantly (about 8 to 10%), if the interlock can absorb about 0.4 to 0.5 of the shearing force created by the load. This is equivalent to a reduction of the section modulus from 91 to 93 per cent. If the interlock absorbs a still smaller share of the shearing force, the extreme fiber stress of the outer fiber does not increase considerably (only to about 115% of the original value) which would mean a total reduction of the section modulus to 87%, if no shearing force is transmitted. The extreme fiber stress of the inner fiber (in the interlock), however, increases more rapidly, if the shearing force drops below 0.4 to 0.5; but in this case it must be considered that for the protected extreme fiber of the interlock a higher stress appears permissible than for the extreme fiber at the back of the pile.

In conclusion, it may be stated that the question of the strength of steel sheet-piling (and, therefore, the question of the section modulus) is often over-estimated as compared with a careful examination of the penetration. One should always observe that, nearly without exception, failures occur, not because the wall is too weak, but because it has not been driven sufficiently deep.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

FLOW OF WATER AROUND BENDS IN PIPES

Discussion

BY DAVID L. YARNELL, M. AM. SOC. C. E.

DAVID L. YARNELL,[26] M. AM. SOC. C. E. (by letter).[26a]—The writer is indeed gratified at the additional data brought out by Messrs. Gumensky, Lansford, and Mavis in their discussions of this paper. Since it was necessary to limit the length of the paper, only the more important findings of the investigation could be presented. The fact that the tests were made on smooth pyralin pipe instead of commercial pipe should not be overlooked. It is quite probable that losses in commercial pipe of the same diameter and radius of curvature would be greater than those obtained in the smooth pyralin pipe.

It was unfortunate that tests could not have been made on pipe of varying radii, but lack of funds prevented such a program. It is a source of satisfaction that the tests reported by Mr. Gumensky as conducted under the direction of R. R. Proctor, Assoc. M. Am. Soc. C. E., have been published in connection with this paper.

The following formulas for loss of head in 6-in. pyralin pipe bends were obtained on the basis of the writer's tests:

For a 45° bend,

$$H = 0.11 \frac{V^2}{2g} \quad \ldots\ldots\ldots\

It will be noted that the paper reports, for a 90° bend, a lower value of K (namely, 0.15), than that used by the Metropolitan Water District of Southern California. On commercial pipe (particularly the larger sizes) the value of K would be considerably greater than 0.15.

When unequal velocity distribution occurs at the beginning of the bend and excessive turbulent flow prevails within, and down stream from it, the loss of head due to the bend and the turbulent flow will occur many feet down stream.

Although it is realized that the writer and the late Professor Nagler did not originate the idea of using a bend as a flow meter, it seemed that this fact had been overlooked by many engineers. Therefore, it was re-emphasized to show that bends in pipes, objectionable as they are, had certain good features. Thus, it is encouraging to study the additional tests by Mr. Lansford on bends as flow meters. Mr. Lansford states that the coefficient is a constant for velocities of 1.5 ft per sec, and greater. This might not be true when unequal velocity distribution prevails at the beginning of the bend. Mr. Lansford's use of a bend as a flow meter in connection with pumps in the Hydraulics Laboratory of the University of Illinois is very unique.

The references listed by Professor Mavis are valuable additions to the paper, As he states, transparent models have done much to throw light on conditions of flow within a pipe or a bend.

The writer wishes to acknowledge the assistance of Mr. Edward Soucek on the computations connected with these tests. In closing, he wants to emphasize what he considers probably the most important finding in the investigation, namely, that it is possible to have such conditions of flow that the loss of head may be very little, or unusually large, in the same bend for identical discharges.

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DISCUSSIONS

STREET THOROUGHFARES
A SYMPOSIUM

Discussion

BY MESSRS. FREDERIC BASS, AND HUGH E. YOUNG

FREDERIC BASS,[14] M. AM. SOC. C. E. (by letter).[14a]—The conclusion is advanced in this Symposium that, in the division of right-of-way costs for street thoroughfares among various beneficiaries, the land values and building costs should be determined separately. This principle is supported by Mr. Herrold on the ground that building costs may be estimated accurately, whereas estimation of land values is likely to be affected by irrelevant influences, and opinions of experts are likely to be supported by insufficient data. Mr. Herrold further states, as a basic principle, that land for streets or for street widenings should be provided by abutting property without cost. In other words, he claims that, in condemnation proceedings, benefits and damages "should be made equal for land acquirements," since the owner receives his remuneration in the increased value of his remaining property and in obtaining the accessibility which his business requires.

It is a fact that when real property is completely transferred, the purchaser assumes all liabilities and encumbrances attached to the property. It is also a matter of common experience that payment for mistakes, whether consciously or unconsciously made, must be equally met. Where it cannot be shown that increased values of remaining property result and accessibility is improved for any particular property, the owner should no doubt be compensated for land taken. Ordinarily, however, the property does increase in value, and accessibility is improved by the street widening, and, to that extent, the owner should contribute. Mr. Herrold's principle appears to facilitate that contribution.

NOTE.—The Symposium on Street Thoroughfares was published in August, 1934, *Proceedings*. Discussion on this Symposium has appeared in *Proceedings*, as follows: October, 1934, by H. Shifrin, M. Am. Soc. C. E., and November, 1934, by Donald M. Baker, M. Am. Soc. C. E.

[14] Cons. Engr.; Prof. of Municipal and San. Eng., Univ. of Minnesota, Minneapolis, Minn.

[14a] Received by the Secretary January 4, 1935.

HUGH E. YOUNG,[15] M. AM. SOC. C. E. (by letter).[15a]—A statement of the principle that land for streets or for street widenings of the standard widths required by the use should be provided by abutting property owners without cost, is an important feature of the paper by Mr. Herrold. His observations and deductions merit the most careful consideration at this time when the development of comprehensive plans for much needed improvements in street transportation systems seems quite remote because of the financial situation arising from dwindling tax resources.

This basic principle is well fortified by fact and logic. It is true, as the author states, that street widths were determined by vision, generosity, cupidity, and probably habit; that the introduction of the motor vehicle has compelled a tremendous increase in the floor area that transportation uses; and that groups of citizens want their streets improved to preserve their business character, to make business houses more accessible, and to prevent mutation.

It is also a fact (see the heading "Users Should Pay") that "in a city the character of use determines the width of the street and the width of the roadway, the latter being based on the almost uniform laws establishing vehicle widths." The roadway widths recommended by Mr. Herrold are standard. As a matter of basic principle it is equitable that the owner of abutting property should pay the major part of the entire cost of the right-of-way or street width required for standard uses of property—the pavement to be paid for as determined by use. This judgment is based, not on the theory that the original dedication was insufficient, because it probably was sufficient at the time it was made, but rather on the theory that added accessibility accruing to the property and the greater value thereby produced is at least an offset to the area of land given up by the owner.

However, as with all principles the mere formula cannot be applied with equal justice in every instance as, for example, where an owner has a piece of property 50 ft deep on an existing street developed with a building adequate for his purposes, and from it 25 ft are taken to widen the street. No theory of inadequate dedication or added value would stand, if the remainder of his land is no longer usable. There is nothing to do in a case like this other than to compensate him for the land taken and damaged.

The basic principle applies to street requirements in a city where the normal use determines the width of the street and the width of the roadway. For arterial streets, express highways, super-highways, and trucking ways, as Mr. Herrold states, the widened right of way must be acquired through special assessment districts or terminals, or other usual proceedings.

Since there are exceptions, it would seem that taking into consideration all the factors (and especially the basic principle that in condemnation proceedings benefits and damages should be made substantially equal for land acquirements), the most practicable plan would be to include under one proceeding land condemnation, damages, construction cost, including side-

[15] Chf. Engr., Chicago Plan Comm., Chicago, Ill.

[15a] Received by the Secretary February 11, 1935.

walks and lighting system—in fact, everything necessary for the completed improvement. A benefit should be assessed against the abutting property in a sum not less than the property damage minus reconstruction costs, and damage claim should be allowed to the remainder, if any. All other costs should be assessed against the municipality or other public body as a public benefit to be paid from gas tax, vehicle tax, or other tax.

For arterial highways, express highways, super-highways, trucking ways, etc., the same principle should be applied except that where the width of the improvement is greater than that required for standard uses of property, the cost of the excess land and building taken should be paid for by the user, or by the public in general, for the obvious reason that the greater width of thoroughfare is for the express benefit of traffic. In other words, all cost in excess of that which ordinarily would be assessed against the abutting property for the standard widening, as outlined by Mr. Herrold, would be assessed against the municipality as a public benefit.

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DISCUSSIONS

SECURITY FROM UNDER-SEEPAGE MASONRY DAMS ON EARTH FOUNDATIONS

Discussion

BY MESSRS. DONALD J. HEBERT, ARTHUR CASAGRANDE, AND CALVIN V. DAVIS

DONALD J. HEBERT,[47] JUN. AM. SOC. C. E. (by letter).[47a]—The recommendation in this paper of a design method for securing dams from under-seepage, based on an exhaustive and systematic compilation of data in existing structures, represents a valuable contribution. With the problem in its present state, the approach toward a solution that has been made by the author is the reasonable and practical one. Until the theoretical aspects of the problem are more clearly understood, an empirical solution is necessary, and the one advanced in this paper is the most satisfactory from a consideration of economy and safe design.

The writer performed some experiments for the U. S. Bureau of Reclamation[48] which he proposes to describe in this discussion, because they have a direct bearing on the subject of the paper by Mr. Lane. The results, which are set forth herein, demonstrate quite clearly the reliability of the present empirical and theoretical solution. Only a part of the results are included, the remainder being filed for reference in Engineering Societies Library, New York, N. Y.

The purpose of the study for which these experiments were performed was to investigate the existing methods of estimating the magnitude of uplift pressures, due to a head of water, under masonry dams on porous foundations. A comparison of the various methods is made on the basis of a collection of observed pressures under a number of existing structures. Pressures at significant points under each structure were computed by each of the design

NOTE.—The paper by E. W. Lane, M. Am. Soc. C. E., was published in September, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1934, by Messrs. William P. Creager, and L. F. Harza; January, 1935, by Messrs. Joel D. Justin, and Louis E. Ayres; and February, 1935, by Messrs. W. M. Griffith, and E. Mckenzie Taylor.

[47] Jun. Engr., U. S. Bureau of Reclamation, Denver, Colo.

[47a] Received by the Secretary, January 22, 1935.

[48] "Hydrostatic Uplift Pressures Under Dams on Pervious Earth Foundations", by Donald J. Hebert, Jun. Am. Soc. C. E., *Memorandum No. 384*, Bureau of Reclamation, U. S. Dept. of the Interior, 1935.

methods, and these values were compared, by means of curves, with the actual pressures measured at the dam. The data as received consisted of readings of water levels in pressure pipes placed in various positions under the dams during construction. These readings were expressed in terms of head lost to the point under consideration. This loss is represented by the difference in elevation between the level in the reservoir and the water level in the piezometer pipe rising from the point in the subsoil. The loss of head was then expressed as a percentage of the total head on the dam. The pressure expressed in this way was assumed to be constant at any point.

From Darcy's law for flow through soil, $V = C_1 \frac{h}{L}$, or $h = \frac{VL}{C_1}$. Since $\frac{L}{C_1}$ is constant for any point in a homogeneous soil, h is directly proportional to the velocity, V, which, in turn, depends on the total head, H, that causes flow. When H increases, V also increases, and there is a greater loss due to friction. Thus, h increases as H increases, keeping the ratio between them theoretically constant. By expressing the loss in this way the effects of fluctuations in head-water and tail-water were eliminated.

Creep distance (which Mr. Lane defines as the distance of travel along the under surface of the dam) was divided into two classes for this study. The creep used in the Bligh method is called plain creep and the creep as introduced in the method advocated by Mr. Lane is designated as weighted creep. All the data were reduced to the same basis and expressed in the same terms. To make the comparison between theoretical and observed pressures a graph was plotted for each dam. The percentage-creep was plotted along the abscissa and the percentage-head loss was plotted along the ordinate. The values at the various observation pipes were connected by straight lines. In order to compute the percentage of plain creep for the horizontal control it was necessary to designate some point as the end of creep. The fact that tail-water potential must be assumed to exist at the end of creep complicated the determination of this point. Due to the presence of weep-holes, long down-stream aprons, grouted and ungrouted rubble masonry, the plane of the tail-water potential, or creep end, could not be located rigidly. An analysis of the conditions at the toe of each dam is included herein accompanied by a sketch of the dam (see Fig. 4). The point at which the end of creep was assumed to occur is indicated on each sketch.

Colorado River Dam.—In the case of the Colorado River Dam, the last pressure pipe was located 29 ft 3 in. up stream from Point *A* (Fig. 4(*a*)). Down stream from this pipe there was no method of determining the pressure. The location of the actual tail-water was not given in the data. Beyond Point *A* the water can escape freely to the tail-water so that a pressure corresponding to the tail-water elevation was assumed to exist down stream from Point *A*. The weep-hole was not taken as the end of creep because it was small and because it was in front of a concrete wall that would tend to create a pressure.

Percha Dam.—Two different locations of creep-end were used in drawing the curves for Percha Dam. In the first case the assumption was made that the rubble concrete, even with the weep-holes through it, offered enough resistance to retain some of the uplift pressure, and the end point was taken at Point *B*

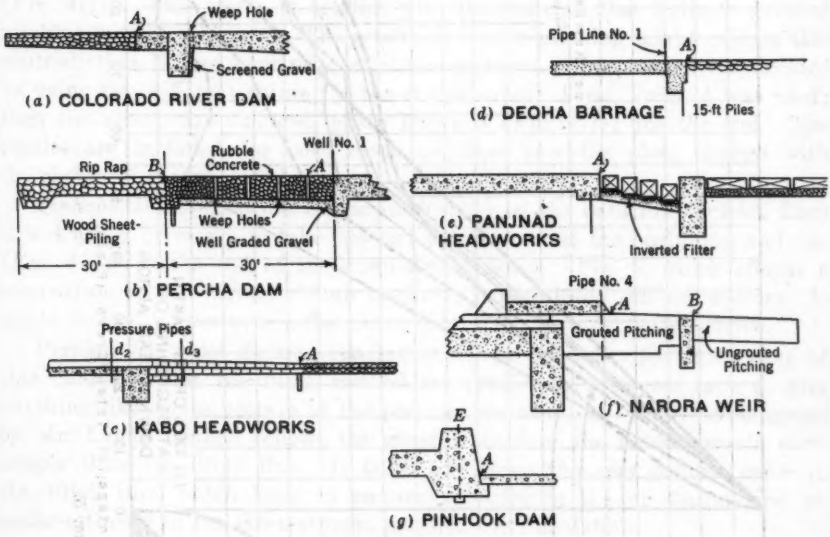

FIG. 4.

(Fig. 4(*b*)). In the second case the first weep-hole was taken as the end point and designated as Point *A* (Fig. 4(*b*)). This is based on the assumption that the weep-hole is large enough to permit the subsoil water to pass through freely.

Kabo Head-Works.—The end of creep at Kabo Head-Works was taken at Point *A*, in Fig. 4(*c*). The original data did not indicate the location of the tail-water potential. The observations of water levels in Pipe d_3 (Fig. 4(*c*)) indicated a residual pressure, and, therefore, the end of the brick pitching was assumed to be the creep-end.

Deoha Barrage.—The end point in the case of Deoha Barrage is rather definitely fixed. The original data included computations which assumed Point *A* (Fig. 4(*d*)) to be at tail-water potential, and, consequently, this point was used.

Panjnad Head-Works.—The beginning of the filter at Point *A* at the Panjnad Head-Works (Fig. 4(*e*)) was taken as 100% creep. A residual head of about 9.4% occurs at this point. It decreased to 7.4% at the end of the filter, but increased again to 8.2% at the beginning of the down-stream rip-rap. This distribution of pressures can be attributed to a dividing of the flow after passing the last line of sheet-piling. The water probably escapes through the filter as well as the rip-rap. The magnitude of the residual pressure can be explained when the assumption is made that the layer of

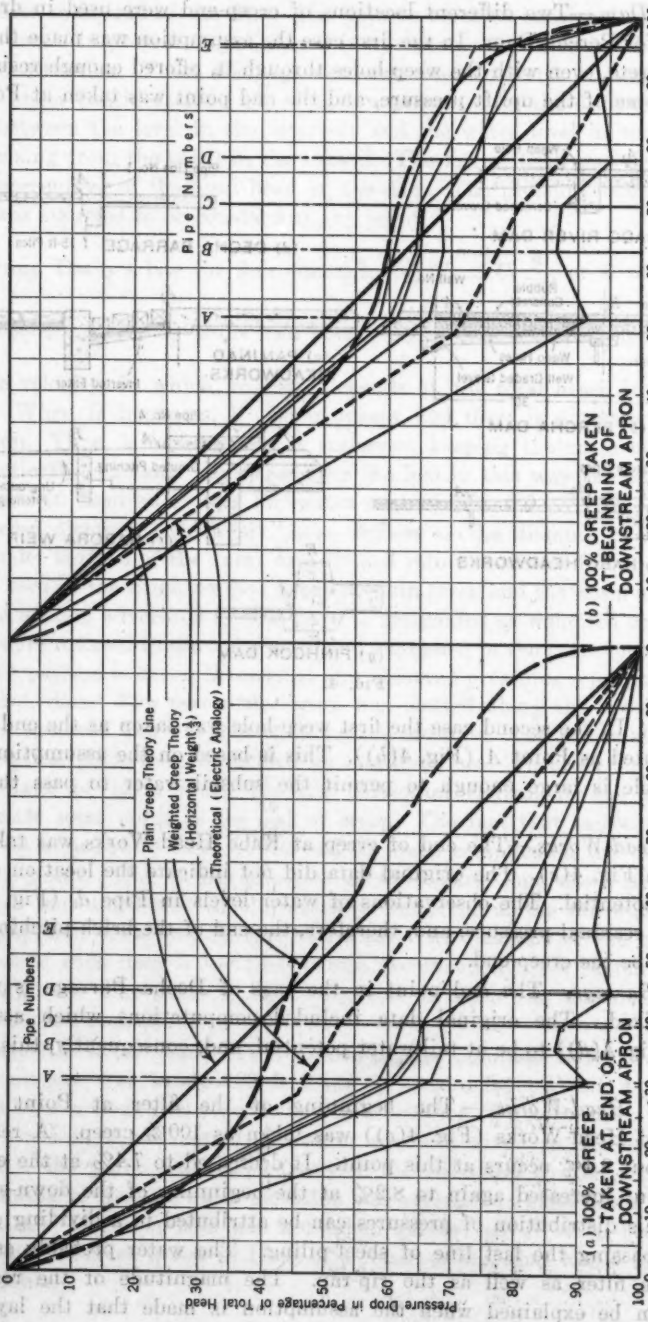
Pressure Drop in Percentage of Total Head
Creep Distance in Percentage of Total Creep (Plain)
Pipe Numbers
E
D
C
B
A
Plain Creep Theory Line
Weighted Creep Theory (Horizontal Weight 1/3)
Theoretical (Electric Analogy)
(a) 100% CREEP TAKEN AT END OF DOWNSTREAM APRON
(b) 100% CREEP TAKEN AT BEGINNING OF DOWNSTREAM APRON

FIG. 5.

clay which is shown on a geological section of other parts of the subsoil extends into the section containing the pressure pipes. This layer of clay would create the pressures indicated.

Narora Weir.—In an analysis which accompanied the data for this weir the assumption was made that the creep at Narora Weir ended at Point *A* (Fig. 4(*f*)). This seems to conflict with the sketches that indicate grouted pitching beyond this point. The condition of this pitching might explain this contradiction, but no description of it was included. The question was decided by using two different points for the end of creep: First, Point *A* was used; then the set-up was changed, using Point *B* (Fig. 4(*f*)) for the end. The results are included for both cases to show how the plots change with the choice of the creep-end.

Pinhook Dam.—Because of the inadequacy of the data for Pinhook Dam it was again necessary to take two end points—one at the beginning and one (Fig. 4(*g*)) at the end of the down-stream apron. Fig. 5, which affords a comparison of the various design methods, is practically self-explanatory. It might be well, however, to point out a few of the features demonstrated.

Perhaps the most definite conclusion drawn from the pressure curve of this dam was that the Bligh method for predicting pressures fails to give anything like a true picture of the pressure distribution. The line computed by Mr. Lane's method follows the general trend of the measurements more closely than the Bligh line. It tends to correct the very definite error in the Bligh line, which leads to an over-estimate in the up-stream, and an under-estimate in the down-stream, parts of the foundation.

ARTHUR CASAGRANDE,[49] ESQ. (by letter).[49a]—In this study of the relationship of the stability of weirs and dams to the percolation of water beneath their foundations, the author has covered so large a number of structures all over the world that the paper represents the most comprehensive study of this kind that has ever been undertaken. For this reason the writer was greatly interested to study this paper in detail, including the unpublished supplement (filed in Engineering Societies Library) containing the cross-sections of most of the structures that were used by the author in his analysis.

Although the writer cannot agree with the general viewpoint expressed by the author, he has found the study of this paper very stimulating. It should certainly be helpful in inspiring engineers to pay more attention to the question of the effect of seepage on the stability of such structures.

Mr. Lane suggests an improvement of Bligh's coefficient of percolation by introducing what he terms "weighted creep." The writer believes that the basic shortcoming of Bligh's method is not eliminated by the introduction of the weighted creep.

In a publication[50] by Charles Terzaghi, M. Am. Soc. C. E., which appeared in 1922, it was demonstrated convincingly by theory as well as by experiments

[49] Asst. Prof. of Civ. Eng., Graduate School of Eng., Harvard Univ. Cambridge, Mass.

[49a] Received by the Secretary January 28, 1935.

[50] "Der Grundbruch an Staumauern und seine Verhütung," von Charles Terzaghi, *Die Wasserkraft*, 1922.

Homogeneous Soil
60 ft
25 ft
20 ft
50 ft
Impervious Stratum
CASE A
Homogeneous Soil
60 ft
25 ft
20 ft
50 ft
Impervious Stratum
CASE B
Graded Filter
Homogeneous Soil
20 ft
60 ft
120 ft
Impervious Blanket
20 ft
50 ft
Impervious Stratum
CASE C
Graded Filter
Homogeneous Soil
20 ft
60 ft
20 ft
50 ft
Impervious Stratum
CASE D
$k' = \sqrt{k_{min} \times k_{max}} = 3\,k_{min}$
k'
k'
25 ft
20 ft
20 ft
50 ft
Transformed Section
k_{max}
k_{min}
$k_{max} = 9 \times k_{min}$
25 ft
Horizontally Stratified Soil
60 ft
20 ft
50 ft
Impervious Stratum
True Section
CASE E

FIG. 6.

that it is the magnitude and distribution of seepage forces near the toe of water-impounding structures (particularly the discharge gradient at the toe) which determines the degree of safety against piping or underground erosion.

In order to judge how misleading the use of Bligh's coefficient of percolation, or any modification of it, really is, compare the five solutions for a low dam on pervious ground shown in Fig. 6. The flow nets were constructed according to the method which Forchheimer developed in 1911.[51] In Table 7

TABLE 7.—QUANTITIES DERIVED FROM THE GRAPHICAL SOLUTIONS IN FIG. 6

Case	Type of soil	Seepage factor	Discharge gradient at the toe	Factor of safety against piping	Effect of scour on factor of safety against piping	Bligh's coefficient of percolation	Lane's weighted creep ratio	Average uplift, in tons per square foot, beneath downstream third of weir	Remarks on Stability; Safety Against:	
									Piping	Uplift
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
A	Homogeneous	0.33	0.22	4.6	Negligible	5.6	3.9	0.33	Safe	Unsafe
B	Homogeneous	0.33	0.44	2.3	Very large	5.6	3.9	0.17	Decidedly unsafe	Safe
C	Homogeneous	0.56	0.11	9.0	Negligible	3.8	1.9	None	Safe	Very safe
D	Homogeneous	0.25	0.05	20.0	Negligible	9.8	3.9	None	Very safe	Very safe
E	Slighty stratified	0.45	0.30	3.3	Negligible	5.6	3.9	0.46	Higher degree of safety recommended	Unsafe

are assembled quantities which were derived from the graphical solutions in Fig. 6 and which are helpful for a comparative study of the effects of seepage on the stability of the structure for the various designs shown.

The seepage loss per unit of length of the dam is found (all quantities measured in a consistent system of units) by multiplying the coefficient of permeability with the difference between head and tail-water level and with the seepage factor, the latter being the ratio of the number of squares between any two neighboring lines of equal head (equi-potential lines) over the number of squares between any two neighboring flow lines. The seepage factor contained in Column (2) of Table 7 may be used, therefore, as a relative measure for the seepage losses of the various cases shown in Fig. 6.

Column (3), Table 7, contains the hydraulic gradient along the flow line at the point of discharge at the toe of the structure, for which the writer is using the term, "discharge gradient." According to Professor Terzaghi's theoretical and experimental investigations,[50, 52] the soil at the toe of the structure will blow up as soon as the discharge gradient reaches a critical value which for average sands is equal to 1.

Column (4), Table 7, contains a factor of safety against piping based on a critical hydraulic gradient of 1. In Column (5) is stated the extent to which the discharge gradient is affected by scouring at the toe of

[51] "Hydraulik", by Phillip Forchheimer, Berlin, 1930 Edition, p. 82.

[52] "Erdbaumechanik", von Charles Terzaghi, M. Am. Soc. C. E., Vienna, 1925, pp. 369–377.

the structure, a factor which should always be considered in the design of such structures. Column (6) contains Bligh's coefficient of percolation, and Column (7), Mr. Lane's weighted-creep ratio. The average hydrostatic uplift on the down-stream third portion of the structure, in tons per square foot, is assembled in Column (8). Columns (9) and (10) contain general remarks as to the stability of the structure as affected by seepage pressures and by hydrostatic uplift.

A comparison of the five cases in Table 7 indicates clearly that neither Bligh's nor Lane's coefficients reflect the degree of safety against piping. For example, Case *B* has only a small degree of safety against piping which would disappear entirely if scouring took place at the toe. Therefore, this design must be considered unsafe. On the other hand, Case *C* has a factor of safety of 9 and scouring has little effect on the degree of safety. However, according to Mr. Lane's weighted-creep ratio, Case *B* appears to be twice as safe as Case *C*.

The examples shown in Fig. 6 and the corresponding data in Table 7 also permit a conclusion regarding the effectiveness of a properly designed filter. It is apparent that such filters beneath the down-stream portion of the structure are more effective in reducing the discharge gradient than an increase in the width of the weir or a sheet-pile wall of average depth. It was Professor Terzaghi[50] who first pointed out the advantages of such filters that are loaded by the weight of the structure itself and are properly protected from scouring action.

From the foregoing, it will be understood why the writer cannot agree with Mr. Lane that a reverse filter is good only for a 10% increase in security.

Under the heading, "Reverse Filters, Weep-Holes, and Drains," the author makes a number of important remarks, conceding that designs of such structures have been used with very short percolation distances. The example of Coon Rapids Dam, in Minnesota, where the coefficient of percolation was reduced accidentally to 0.6 without ill effects, shows strikingly the effectiveness of loaded reverse filters as a method for preventing piping as well as unfavorable uplift pressures. The writer wishes to emphasize that there is no other way of knowing whether or not a filter and drains will be effective than by Professor Terzaghi's theoretical and experimental approach to these problems.

During the ten years, 1925–35, Professor Terzaghi and others have shown that the average permeability of a natural deposit of soil in a horizontal direction (k_{max}) is always larger than that in a vertical direction (k_{min}); and that $k_{max} = nk_{min}$, in which n varies between 2 and 10. For distinctly stratified soils the value of n may become much larger than 10. Such soil conditions reduce the safety of a structure if compared with the safety of a similar structure on homogeneous soil with an average permeability, $k' = \sqrt{k_{max} \times k_{min}}$, which will yield the same amount of seepage. As an example, consider Case *A* in Fig. 6, and assume that the average coefficient of permeability of the soil in a horizontal direction, k_{max}, is nine times greater than k_{min} for the vertical direction. As a result of the anisotropic character of the soil, the flow net will be distorted as shown in Case *E*, Fig. 6

Flow lines and lines of equal head do not intersect at right angles, except for certain positions, and the squares are elongated in the horizontal direction. For the construction of this flow net, the writer used a method which was originally developed and applied to seepage problems by A. F. Samsioe, of Stockholm, Sweden, in 1930. According to this method, one can find the actual flow net for an anisotropic soil by plotting an ordinary flow net in a transformed cross-section of the structure in which all horizontal dimensions are reduced by the factor, $\sqrt{\frac{k_{min}}{k_{max}}}$. In the present example, this factor is $\frac{1}{3}$. The transformed section and flow net are shown in Case *E*, Fig. 6. After reversing the transformation, the actual shape of the flow net is as shown in Case *E*, "True Section."[53]

A comparison of the structure on homogeneous soil (Fig. 6, Case *A*) with the same structure on slightly stratified soil (Case *E*), is afforded by the data contained in Table 7. The stratification has increased the seepage losses by about 30% and decreased the factor of safety by approximately the same amount. For distinctly stratified soils, the discharge gradient may easily reach the critical value. This example demonstrates the tremendous importance of stratification of the soil and of other "minor geologic details"[54] on the safety of such structures against piping.

Mr. Lane's statistics reflect the effect of such stratification and to a certain extent he succeeds in refining Bligh's method in the right direction. For example, in Case *A* (Fig. 6), if one wished to increase the safety by increasing the horizontal path or the vertical path of percolation, Bligh's coefficient would indicate no preference, while Mr. Lane's method gives preference to the vertical faces of a sheet-pile wall. However, the ratio for this preference is fixed instead of being dependent on actual soil conditions. In addition, the "weighted creep" as well as Bligh's coefficient disregard entirely the fundamental importance of the relative position of vertical faces to the toe of the structure, for which there is no other method of analysis available than that developed by Professor Terzaghi.

The writer believes that any statistical approach to this problem as attempted by Bligh and the author can be defended only as long as the mechanics of piping is not properly understood and as long as there are no scientific methods available to analyze seepage pressures and particularly their effect on the stability of the soil near the toe of such structures.

The safety of a structure designed on the basis of Bligh's or Lane's coefficients of percolation is subject to the law of probability on which these methods are based, thus meaning that in most cases the structure will be many times over-designed, whereas occasionally local soil conditions, perhaps combined with an unfavorable cross-section, will invite failure. However,

[53] A general derivation for this transformation formula is contained in a paper by Professor Schaffernak, entitled "Erforschung der physikalischen Gesetze, nach welchen die Durchsickerung des Wassers durch eine Talsperre oder durch den Untergrund stattfindet," *Die Wasserwirtschaft*, 1933, No. 30.

[54] "Effect of Minor Geologic Details on the Safety of Dams," by Charles Terzaghi, M. Am. Soc. C. E., Am. Inst. of Min. and Metallurgical Engrs., *Technical Publication No. 215*, February, 1929.

Professor Terzaghi's analysis of the seepage forces, if based on extensive investigations of the soil conditions, does provide a satisfactory basis for choosing a safe and economical cross-section which will best fit the peculiarities of the foundation conditions in each case.

The general character of, and any uncertainties in, the soil conditions can easily be taken care of by choosing an appropriate factor of safety. The writer believes that the factor of safety against a blow-up at the toe should be at least 3 for homogeneous soil strata with a coefficient of permeability, k_{max}, of not more than two or three times k_{min}, and with a sufficiently high vertical face at the toe so that scouring will have only a small effect on the magnitude of the discharge gradient. However, if the soil conditions are not sufficiently explored to permit reliable conclusions regarding their uniformity, or if such exploration and soil tests indicate the presence of stratified or irregular deposits (particularly if consisting of uniform fine-grained sands) the factor of safety should be at least 10, and still larger for very uncertain foundation conditions. It was on such a basis that the writer arrived at the concluding remarks contained in Columns (9) and (10) in Table 7.

In conclusion, the writer wishes to state that the study of Mr. Lane's paper has fortified his belief that Professor Terzaghi's theory on piping and underground erosion and his methods for analyzing the stability of weirs and dams are in excellent agreement with practical experience and that they also represent the only existing approach to these problems which is satisfactory from a scientific standpoint. Their application is strongly recommended in the design of new structures, as well as in the analysis of the stability of many existing weirs and dams.

CALVIN V. DAVIS,[55] M. AM. SOC. C. E. (by letter).[55a]—An exhaustive research, which covers the characteristics of a considerable number of masonry dams on soft or porous foundation, forms the basis of this valuable and timely paper. The conclusions, therefore, have been crystallized from the practical experiences of the very limited group of engineers who have labored in this highly special field. As experience is by far the most important factor in the design of such structures, these conclusions are entitled to both respect and thoughtful consideration. No better method can be found as a starting point for the creation of a new method of analysis of creep-flow ratios than a thorough examination of the record of as many existing dams of reasonable height as may be available and the requirements of design that were followed.

The results of the author's comprehensive survey have been expressed in terms of a modified form of Bligh's ratio in which the vertical surfaces of cut-off walls are assumed to have three times the resistance to water travel possessed by the under sides of horizontal floors and hearths. Although Mr. Lane's research confirms the fact that the resistance of vertical surfaces to flow is greater than that of many forms of horizontal surfaces, the results are not conclusive enough to give each a definite weight. Actually, the rela-

[55] Chf. Designer, Ambursen Dam Co., Inc., New York, N. Y.
[55a] Received by the Secretary February 26, 1935.

tion between the resistance to water travel of horizontal-creep and vertical-creep surfaces will vary with both the foundation material and the type of structure built on it.

It is certain that under conceivable conditions the resistance of a horizontal foundation slab might be considerably less than one-third that of the vertical surfaces of a good cut-off wall. If, however, a foundation were serrated as shown by Fig. 7, the under side of the floor should have a water-travel resistance equal to that of a vertical cut-off surface. The floor of the dam shown by Fig. 7 would be tightly pressed against the foundation material by the inclined components of buttress stress, which are considerably greater in magnitude than the vertical components acting on a horizontal floor. Sliding is also partly resisted by compressive and shearing forces rather than by friction, and this also would have a tendency to tighten the foundation material against the floor. It would not be necessary for the inclined surfaces of this floor to be as steep as 45° to be as effective as vertical surfaces, as suggested by Mr. Lane.

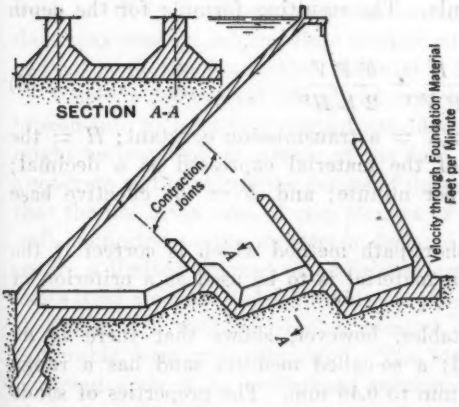

FIG. 7.

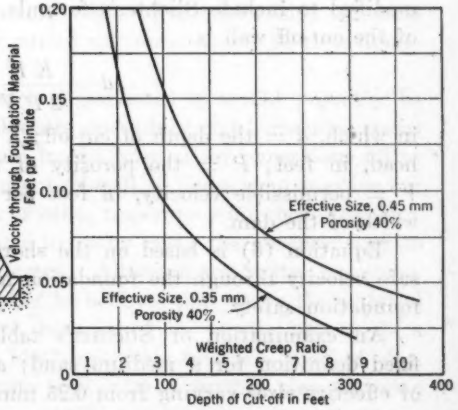

FIG. 8.—HEIGHT OF DAM, 75 FEET; EFFECTIVE HORIZONTAL CREEP, 45 FEET; FOUNDATION MATERIAL, MEDIUM SAND.

The writer has prepared various designs for this type of construction for specific projects, varying from 25 ft to 210 ft in height, and for different kinds of earth foundations. Although as yet none has reached the construction stage, a careful analysis of the designs indicated that this type possesses considerable merit, particularly for high dams in which the introduction of a contraction joint along the trajectory of first principal stress may be indicated. Its resistance to creep is obvious.

The uncertainties connected with the theoretical basis of the investigation as developed by the author from Darcy's law do not allow fixed weighted-creep ratios to be established with consistency. In applying Darcy's law as expressed by the author in Equations (1), (2), and (3), the required length of creep may be determined by the direct method of applying coefficients

which experience has proved to be amply safe—or by assuming a velocity of flow through the foundation material that would appear to give a large margin of safety against piping failure—and then determining the depth of cut-off required to hold to this velocity. The latter method is commonly based on the formula derived by Professor C. S. Slichter which was also referred to by the author.

A comparison of these two different approaches to the same problem reveals a decided lack of agreement. Typical examples will illustrate this discrepancy.

Assume a spillway dam of the Ambursen type on continuous floor, founded on medium sand, having a height of 75 ft and a total horizontal creep of 135 ft. The depth of cut-off required for this dam by the author's method (adopting his recommended weighted-creep ratio of 6) would be:

$$d = \frac{1}{2}\left((6 \times 75 \text{ ft}) - \frac{135}{3}\right) = 202.5 \text{ ft.}$$

In determining the cut-off depth by the second method, Darcy's law is modified to include Slichter's formula. The resulting formula for the depth of the cut-off wall is:

$$d = \frac{K H}{2 P V} - \frac{b^2 P V}{2 K H} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

in which, d = the depth of cut-off; K = a transmission constant; H = the head, in feet; P = the porosity of the material expressed as a decimal; V = permissible velocity, in feet per minute; and, b = the effective base width of the dam.

Equation (6) is based on the short-path method which is correct if the safe velocity through the foundation material is to be used as a criterion of foundation safety.

An examination of Slichter's tables, however, shows that there is no fixed definition for a medium sand; a so-called medium sand has a range of effective sizes varying from 0.25 mm to 0.45 mm. The properties of sands having this variation in size are by no means the same, and yet by arbitrarily using the fixed-creep ratio of 6 recommended by Mr. Lane the same treatment would be given to all the foundation materials coming within this classification.

To illustrate, Fig. 8 shows the relation (based on Equation (6)) between depth of cut-off, velocity, and weighted-creep ratio for the dam in the foregoing problem if two sands each having a porosity of 40% and effective sizes of 0.35 mm and 0.45 mm, respectively, are assumed. It will be observed from Fig. 8 that the weighted-creep ratio for the finer material need only be about 62% of that required for the coarser material in order to give the same velocity through the foundation materials or equal theoretical factors of safety against failure by piping.

A velocity of 0.2 ft per min is considered by Joel D. Justin,[58] M. Am. Soc. C. E., and other authorities to be ultra safe for a medium sand. A

[58] "Earth Dam Projects," by Joel D. Justin, John Wiley & Son, 1932.

weighted-creep ratio of 6 gives a depth of cut-off 2.7 times that required to give the velocity through the coarser material and 5.0 times that required to give it through the finer material.

The depth as determined from Slichter's formula, however, is based on the short-path flow. The results obtained in the foregoing computations in one respect substantiate Mr. Lane's conclusion as to the weakness of the short-path theory. Actually, a much greater length of creep is necessary than that required to give merely a safe velocity through the underlying foundation material. Both experience and theory in this case also indicate that there is a greater probability of seepage following the plane of contact of the dam with the foundation material.

The writer has found, however, that the length of water travel need not be as great as required by the weighted-creep ratios recommended by the author if proper construction methods are followed. His experience with the design of permeable foundation dams has been that lower ratios may be used with safety if the foundation structures consist of continuous, heavily reinforced, concrete slabs flexibly connected to substantial vertical cut-off walls at their up-stream and down-stream ends. The under side of these slabs may contain intermediate shallow cut-off walls and should be in intimate contact with the foundation material.

Where a very great length of travel is indicated it would probably be broken into at least two vertical cut-off walls, and additional horizontal travel might be installed up stream of the dam in the form of an impervious fill or a concrete mat to reduce the depths of the cut-off wall. There is no question but that the use of an area of clay blanket or other impervious fill above the dam, well connected to the up-stream deck, is in many cases the cheapest way to secure a limited amount of additional creep. Although the writer has sometimes used a concrete slab for this purpose, he believes that where the material is available a well-placed clay blanket, because of its plastic nature, is a preferable form of construction. The ideal up-stream mat is a layer of clay, carrying upon it an articulated concrete slab, with a further clay fill upon the concrete.

Empirical designing rules and theoretical assumptions must always have their part in dam design, but lack of care or skill in actual construction may be more dangerous to the stability of a dam than errors in designing assumptions. The writer fully agrees with the soundness of Mr. Lane's judgment in preferring a concrete cut-off properly built in contact with the sides of the trench to a steel or timber sheet-piling. The writer would use the latter in any case, except where the depth to which it is to be driven is relatively small and the material one susceptible of easy penetration. Wandering, brooming, and splitting is the rule rather than the exception for Wakefield, splined, or tongue-and-groove sheeting.

In the case of steel sheet-piling, its advantages are fully realized, but its uncertainties in hard driving are also recognized as well. Furthermore, the writer has never been satisfied that the use of such piling in certain types of foundation materials may not reduce the resistance to water travel

by inviting pipes along its smooth sides, particularly along the lines of the structural irregularities necessary to its interlock. The use of jets for sinking difficult cut-off piling may well impair its creep-resisting value in many materials through displacement of well-bedded particles and the disturbance of all the immediately adjacent material.

In the case of the conventional concrete cut-off wall, when constructed in almost any material other than hardpan, well-cemented gravel, or very stiff clay, it is more often than not subject to faulty construction and its creep value may even be less than good horizontal creep. These field defects are inherent in wet foundation construction where men are working at a disadvantage and where flowing water will carry with it the finer materials adjacent to temporary sheeting. Even if performed in the dry, if sheeting and bracing is necessary, the withdrawal of the sheeting invites displacement of foundation material and faulty contact between the concrete and the sides of the trench. The writer recalls a recent example of the human errors which may nullify the most careful design. Despite the specifications, a cut-off into fractured shale, instead of being poured solidly into the trench, was neatly formed. The remaining area was back-filled with broken shale, reducing the value against creep to the bare thickness of the wall at the bottom where it contacted the seamed and partly disintegrated shale. In another and older case, a cut-off wall was built in gravel, boulders, and sand, and the up-stream part of the trench was actually back-filled with construction débris as the sheeting was withdrawn. Early failure of this structure was readily predicted by the designer unless the trench were re-excavated and a puddle fill substituted; it occurred a month or two later while the city officials were still meditating on his recommendations.

Many attempts to grout porous foundation materials have convinced the writer that improvement in the value of horizontal creep cannot be assured by this method, as suggested by the author. Operative difficulties limit the value of grouting beneath a poorly constructed foundation slab. Perhaps some of the methods of chemical impregnation or permanent gelatinization may yet be developed to the point where suspected areas may be closed against percolating water, but the conventional cement grout has been tried many times and although found to be helpful in some cases, limited reliance on its efficacy is advised.

For horizontal creep, the writer has always advised against placing too much reliance upon long hearths below the dam, whether monolithic or articulated, but particularly the latter. These offer little, if any, resistance to the underflow of water and are entitled to slight consideration in determining the proper depth to which the vertical cut-off should be carried. An extensive program of experimental research is urgently needed to establish fundamental principles.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

UPLIFT AND SEEPAGE UNDER DAMS ON SAND

Discussion

BY MESSRS. HARRY H. HATCH, DONALD J. HEBERT, AND C. A. MOCKMORE AND JOHN W. DOUGHERTY

HARRY H. HATCH,[30] M. AM. SOC. C. E. (by letter)[30a].—Undoubtedly there is still considerable to be done in laboratory and field research, especially in obtaining dependable data from actual structures. The author should be commended for calling attention to a new field of investigation for the safety of masonry dams built on sand foundations.

One more factor may be added to those cited by the author effecting the value of the coefficient of permeability, K, in Equation (1), namely, the shape or the slope of the gradation curve below the effective size limit. It is known that the flatter the slope of this section of the gradation curve the smaller will be the value of K. Sample materials of the same physical characteristics with the same effective size and with almost identical particle sizes and gradation curves above the effective size limit, but with different slope gradation curves below the effective size, showed a variation in the values of K of as much as 300 per cent.

Confining his study to sand, Mr. Harza narrows the field of application of this important investigation. What is his definition of sand? In other words, what is the lower limit of the particle sizes of sand considered by the author? Why not extend the analysis so that the study may embody soils in general?

Soil is being studied with special interest from almost every angle; and appreciable progress has been made and considerable information is available on the subject. Had the author proposed his study for soils in general, it is believed that the principles advocated would have had a broader field of application, and that the new wealth of knowledge pertaining to the science of soils could be used to advantage. The principle introduced in the paper has a broader application and usefulness than if it were confined to sand alone.

NOTE.—The paper by L. F. Harza, M. Am. Soc. C. E., was published in September, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1934, by Messrs. Edward Godfrey, and Hibbert M. Hill; and January, 1935, by Messrs. Joel D. Justin, and Charles Terzaghi.

[30] Engr. in Chg., Cobble Mountain Reservoir, Springfield Water-Works, Westfield, Mass.

[30a] Received by the Secretary January 17, 1935.

DONALD J. HEBERT,[31] JUN. AM. SOC. C. E. (by letter).[31a]—The theory upon which the electric analogy method is based, has been adequately discussed in this paper. Before any attempt is made to adopt it as a basis for design, as Mr. Harza has attempted to do, an examination of the many curves should be made. The writer has found[32] that the line representing the pressures as predicted by an electric analogy analysis is for the most part no closer to the observed values than the line obtained by weighted creep computations. The discrepancy between actual and predicted pressures in the case of the analogy is particularly evident in the down-stream part of the under profile. Some of this discrepancy may be due to the uncertainty with which the end of creep was located. In this region the use of the Bligh method is questionable because it under-estimated the actual pressures. The electric analogy, on the other hand, might lead to an unnecessarily heavy and uneconomical design.

Before the electric analogy method can be used as a basis for rational design, it needs to be extended considerably. The technique of the method must permit at least an approximation of the various geological features of any particular subsoil. As a prelude to such an extension of the analogy technique, there is need for considerable progress in the methods for exploring the subsoil and for locating the various features which affect the flow of water through it. Charles Terzaghi, M. Am. Soc. C. E., has emphasized[33] this need by pointing out the possibility that certain geological features, known as "minor details," are often factors which determine that stability of a structure when built on a porous foundation.

Until the point is reached at which it is possible to investigate all the geological features of a subsoil and to reproduce them in an electrical set-up, the analogy method must be used with caution.

C. A. MOCKMORE,[34] M. AM. SOC. C. E., AND JOHN W. DOUGHERTY,[35] JUN. AM. SOC. C. E. (by letter).[35a]—Since this paper appeared the writers have been interested in the verification of the formula (Equation (6)) given by Mr. Harza for the "flotation gradient." In keeping with the suggestion offered in the "Synopsis" that "these theoretical laws are to be used as scientific nuclei around which to build future experimental data," the writers have been making some experiments at the Oregon State College with water flowing vertically through beds of different kinds of granular materials, and, in general, have succeeded in finding this formula valid.

Mr. Harza cites Professor Terzaghi's experiments in which the measured internal pressure to cause flotation always equalled or exceeded the theoretical by ratios varying from 1.0 to 1.24. This, the writers have been unable to verify, except as will be shown subsequently in the case where the granular material was lead shot.

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[31] Jun. Engr., U. S. Bureau of Reclamation, Denver, Colo.

[31a] Received by the Secretary January 22, 1935.

[32] *Proceedings*, Am. Soc. C. E., March, 1935, Fig. 5, p. 364.

[33] *Technical Publication No. 215*, Am. Inst. of Min. and Metallurgical Engrs.

[34] Head of Civ. Eng. Dept., Oregon State Coll., Corvallis, Ore.

[35] Bonneville Dam Section, U. S. Engr. Office, Portland, Ore.

[35a] Received by the Secretary February 19, 1935.

A diagrammatical sketch of the apparatus used by the writers to simulate conditions described in Mr. Harza's Fig. 3 (*d*) is presented in Fig. 25. The same apparatus is shown in operation in Fig. 26. The sides of the box

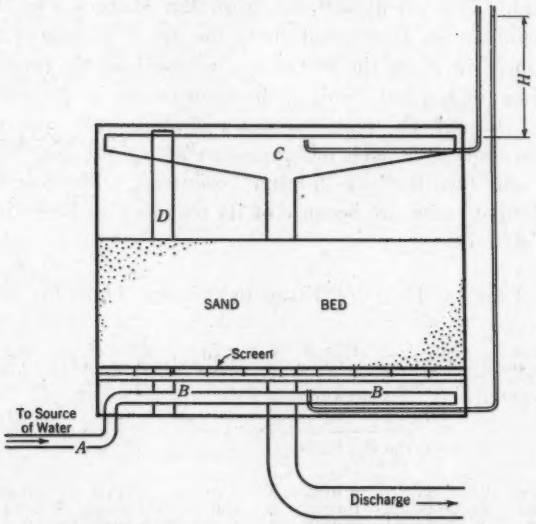

FIG. 25.—APPARATUS FOR DETERMINATION OF LOST HEAD IN FLOW OF WATER THROUGH SAND.

containing the sand were made of glass to assist in the study of the sand movement. In Fig. 25, the water entered at Point *A* and spread into a series of pipes, *B*. These pipes were formed in a series of three rows across the length of the tank and were perforated with small holes to permit even distribution of the water over the bottom of the tank in the space under the sand. The water could then flow vertically upward through the bed of sand and proceed to the drains, *C* and *D*. The overflow, *C*, had a weir crest length of approximately 4 ft, and was arranged so that the water could be discharged through a vertical pipe which led to a weighing scales. Drain *D* was merely an overflow pipe to care for excess flow beyond the capacity of Drain *C*.

The characteristics of the samples and the test set-up are listed in Table 4. The sample to be tested was supported by a 50-mesh screen which

TABLE 4.—CHARACTERISTICS OF TEST SAMPLES

Definition	Sample No. 1, white filter sand	Sample No. 2, Columbia River sand	Sample No. 3, Willamette River sand	Sample No. 4, lead shot	Definition	Sample No. 1, white filter sand	Sample No. 2, Columbia River sand	Sample No. 3, Willamette River sand	Sample No. 4, lead shot
(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
Average specific gravity	2.59	2.69	2.59	11.36	Area of bed, in square feet	1.97	1.97	1.97	0.196
Porosity (a percentage)	38.9	37.6	32.8	43.2	Water temperature, in degrees Fahrenheit	50	52	52	0.1733
Depth of bed, in inches	6.7	6.4	7.8	3.5					

rested in turn on a 4-mesh screen. This 4-mesh screen rested upon heavy tin strips, $\frac{1}{2}$ in. high and extending across the box. The piezometers shown in Fig. 25 accorded an easy means of measuring the loss of head in flow through the sand. This set-up differed from Mr. Harza's Fig. 3 (*d*) in that his piezometer connection, *B*, was not above the upper surface of the sand bed.

With the sample in place the water was released slowly into the box, and at various readings of the lost head, *h*, the time necessary for 100 lb or 200 lb of water to flow through the material was recorded. The quantity of water flowing was increased after each measurement of weight and time, until the sand expanded and "boiling" or "piping" occurred. The lost head at this instant was difficult to measure because of its tendency to lower just as piping occurred (see Table 5).

TABLE 5.—Loss of Head in Flow of Water Through Sand Beds

Head loss, in inches (1)	Flow of water, in cubic feet per second (2)	Head loss, in inches (1)	Flow of water, in cubic feet per second (2)	Head loss, in inches (1)	Flow of water, in cubic feet per second (2)	Head loss, in inches (1)	Flow of water, in cubic feet per second (2)
(*a*) Sample No. 1		(*a*) Sample No. 1 (*Continued*)		(*c*) Sample No. 3		(*d*) Sample No. 4	
1.2	0.0018	5.35	0.0195	2.60	0.0069	1.92	0.0182
1.8	0.0031	5.45	0.0216	3.45	0.0096‖	3.42	0.0256
2.9	0.0052	5.50	0.0258	4.15	0.0119	4.77	0.0317
3.5	0.0064	(*b*) Sample No. 2		4.85	0.0145	6.72	0.0386
4.0	0.0073			5.30	0.0170¶	9.54	0.0475
4.5	0.0083			5.80	0.0200	12.11	0.0534
4.9	0.0092	1.8	0.0340	6.30	0.0219	13.92	0.0582
5.25	0.0098	3.3	0.0561	6.50	.....*	17.16	0.0641
5.5	.....*	4.2	0.0653	6.85	0.0276	18.11	0.0682
5.2†	0.0093	4.7	0.0711	6.85‡	0.0333	21.00	0.0697*
5.45	0.0118	4.9	0.0745	6.85§	0.0390	21.60	0.0763
5.45	0.0149	5.2	.....*	6.85	0.0470	21.50	0.0942
5.45	0.0172	....	.....	6.85	0.0695	23.05	0.1335

* Boiling began. † Dropped to 5.2 with continued boiling. ‡ Dropped to 6.85 from 7.2.
§ Dropped to 6.85 from 7.4. ‖Small movement of fines. ¶ Good separation completed.

Mr. Harza mentions the possibility of conclusions, reached by testing homogeneous materials, which do not apply to the conditions in practice where foundation materials are not homogeneous. In fact, under laboratory control the conditions are none too good even when the sand sample is fairly well graded, and has had a chance to be separated by the flowing water.

The writers thought it best to make sieve analyses of the several granular materials after they had been tested in the apparatus. This would wash out the dust and fine clay that might be there, but it would also tend to cause separation of the fines from the coarser materials and make sampling for sieve analysis more difficult. However, with as much caution as possible after each test for loss of head, a 2-kg or 3-kg sample of the material was taken and dried. This dried material was used in the American Society for Testing Materials standard tests for the determination of the apparent specific gravity and the approximate percentage of voids in inundated fine aggregate. The same tests were used in the case of the lead shot, although the tests are not actually standard when such a heavy material is used.

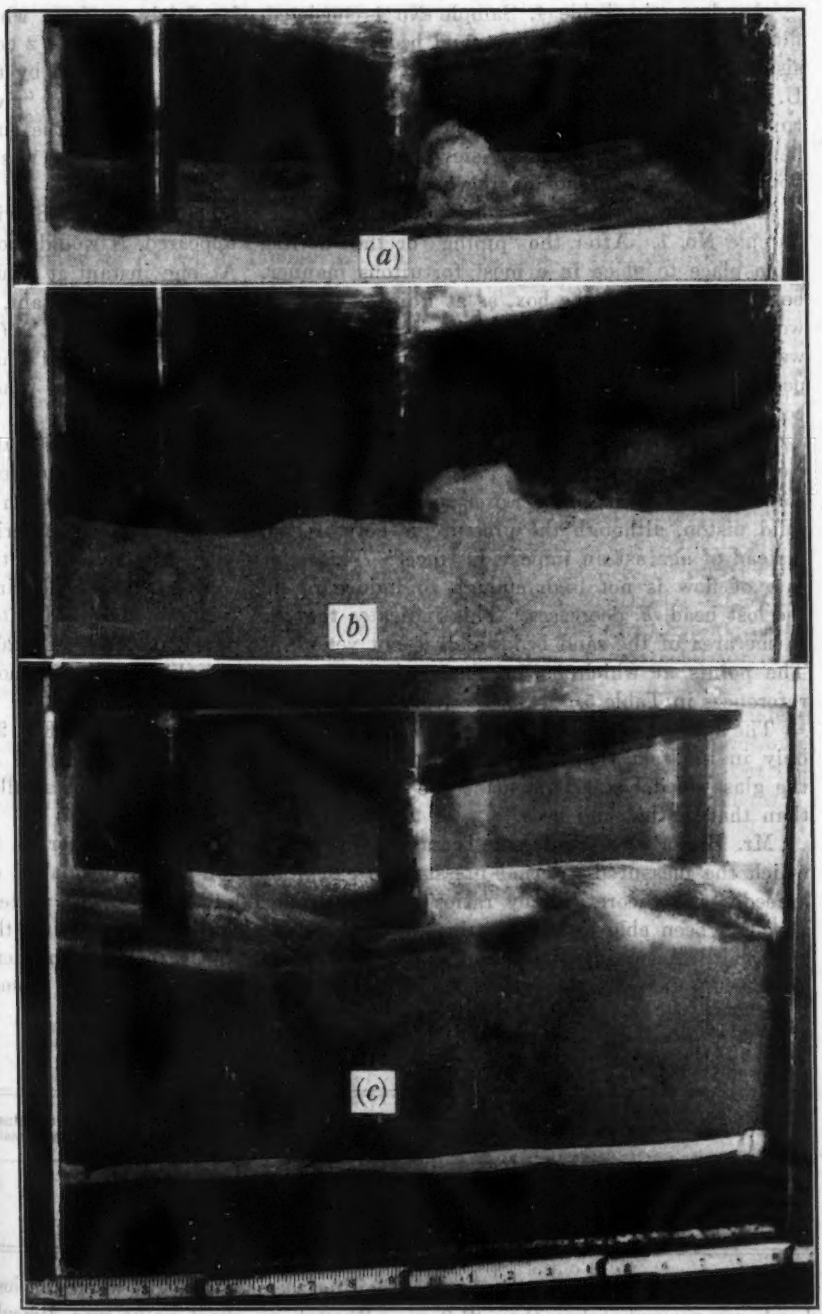

FIG. 26.—FORMS OF "PIPING" WHEN WATER FLOWS VERTICALLY THROUGH SAND BED.

As shown in Table 4, Sample No 1 consisted of a fairly uniform white filter sand commonly used in sand beds for water filters. Sample No. 2 consisted of a fairly uniform Columbia River sand which was furnished by the U. S. Engineer Office, at Portland, Ore. Sample No. 3 consisted of "river run" Willamette River sand from which nearly all material except the sizes between the 6 and 28-mesh sieves was eliminated. Sample No. 4 was a nominal "BB" lead shot having an average diameter of 0.1733 in.

The views in Fig. 26 were taken when the test box was loaded with Sample No. 1. After the "piping" or the "boiling" appeared, it would move from place to place in a most fortuitous manner. At one instant it would be in the center of the box, as at Point *A*, in Fig. 26; in another instant, it would migrate to some such place as Point *B*, or Point *C*. When Fig. 26(*c*) was taken, the boil was moving toward the right. The operator's hand could penetrate the sand easily, as if it were in suspension, while the boiling would continue.

Point is made of this phenomenon in connection with Mr. Harza's statement that "the conclusion is justified that under all conditions" the increment force exerted by seeping water against a granular material acts "as on a solid piston, although the pressure is actually applied through the material instead of against an impervious face." This is evidently true as long as the rate of flow is not high enough to induce piping. After "piping" begins, the lost head, h, decreases. This indicates that the "piston" effect is limited to the area of the sand bed which is not yet piping or boiling (see Fig. 26). The points at which boiling began in each case, is indicated by footnote references in Table 5.

The apparatus for testing the lead shot differed from that shown in Fig. 26, only in the fact that a 6-in. transparent pyralin pipe was used instead of the glass-sided box. This made the cross-section of the bed of shot smaller than that of the sand beds.

Mr. Harza cites Professor Terzaghi's experiments on varied materials in which the measured internal pressure to cause flotation "always equalled or exceeded the theoretical, by ratios varying from 1.0 to 1.24." The writers have not been able to secure values as high as 1.0, except in the case of the lead shot. Assuming the actual gradient as h, and the theoretical gradient as $(1 - P)\ (s - 1)$, the ratios should be about as shown in Table 6. Some-

TABLE 6.—RATIOS OF MEASURED INTERNAL PRESSURE TO THEORETICAL PRESSURES

Material	$\frac{h}{l}$ (Actual)	$(1 - P)\ (s - 1)$ (Theoretical)	Ratio of actual to theoretical
White filter sand	0.814	0.97	0.840
Columbia River sand	0.812	1.05	0.774
Willamette River sand	0.878	1.07	0.822
Lead shot	6.000	5.88	1.020

times in the case of the lead shot, the boiling seemed to begin when the lost head was somewhat less than 21.0 in., although an exact value was difficult

to determine. If the lost head were actually 20.5 in., instead of 21.0 in., the ratio, $\frac{h}{l}$, would become 5.80, which checks very closely with the value for $(1 - P)\ (s - 1)$.

The data given in Table 5 may be shown graphically, if ratios of the actual to the theoretical flotation gradients are used as ordinates, and the rate of flow per square foot of sand bed is used as abscissas. When this was done (curves not published) the point at which "flotation" changed to "piping" was found to be rather indefinite. For Sample No. 2 the head, h, rose to 7.2 in. before "piping" began, and then it dropped to 6.85 in. This same phenomenon did not appear when the lead shot was used. From these experiments it appears that the ratio of the measured to the theoretical head to cause flotation would be less than 1.0, rather than the higher value cited by Mr. Harza.

While testing Sample No. 3, which was not a very uniform material, it was noticed that at the incipient stage of flotation the finer portions of the sand would be raised to the top of the bed; then "boiling" or "piping" would take place in this fine aggregate before it would in the coarser material. Vigorous boiling would keep the sand bed quite well mixed, but the incipient stage of flotation would permit the smaller grains of sand to be raised above the larger. If this is compared with the critical conditions that Mr. Harza describes in Fig. 5 as cross-flow toward the easiest point of escape at the toe of a dam, it seems reasonable to conclude that this fine material might be easily washed away.

In making the tests described in this discussion, the writers have endeavored experimentally to verify the validity of Mr. Harza's "flotation" formula, Equation (6). In general, it is believed that the formula is correct, but that considerable care should be exercised in applying it to practical problems.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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DISCUSSIONS

AN ASYMMETRIC PROBABILITY FUNCTION

Discussion

BY L. STANDISH HALL, ASSOC. M. AM. SOC. C. E.

L. STANDISH HALL,[30] ASSOC. M. AM. SOC. C. E. (by letter).[30a]—This paper follows a series of papers, covering the same subject, which have been presented to the Society from time to time. Considering the mathematical nature of the subject-matter, the author has handled the text in a very readable manner.

The curves presented are of a sufficiently flexible nature to fit a given set of data more satisfactorily than any other yet prepared. This is particularly true with regard to maximum and minimum values. However, from the standpoint of logic, rather than statistics, it is evident that many short records of hydrological functions cannot be expanded by any formula to give a reliable indication of the probable happening in a 100-yr or 1 000-yr period.

The author anticipates this difficulty for he states (see "Synopsis") that, "unless the data are very complete, the final results will seldom have meaning because of the magnitude of the probable errors of the parameters involved." In discussing end points he concludes, furthermore, that it may be necessary to "set limits to the problem from extra-statistical considerations." These two quotations state clearly the nature of the difficulty encountered in applying statistical methods to short records. Strictly speaking, it is the writer's opinion, after careful analysis of many records, that the distribution of data cannot be determined with accuracy for a 100-yr period until at least a 200-yr record is available.

The point in question can be best illustrated perhaps by an example taken from the record of the seasonal run-off of the Mokelumne River in California. A record (partly estimated from neighboring streams) is available for a period of 39 yr, from 1895–96 to 1933–34. A duration curve drawn from these data is shown in Fig. 4(*a*). This is the curve of natural run-off from 1895–96 to the season, 1933–34, measured and estimated. The mean annual run-off for the 39-yr record is 744 000 acre-ft; but a consideration of rainfall

NOTE.—The paper by J. J. Slade, Esq., was published in October, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: January, 1935, by Messrs. Gordon P. Williams, and H. Alden Foster; and February, 1935, by Messrs. R. D. Goodrich, and F. T. Mavis.

[30] Chf. Hydrographer, East Bay Municipal Utility Dist., Oakland, Calif.

[30a] Received by the Secretary January 4, 1935.

records, which cover about twice the period of time, indicates that the mean seasonal run-off of the Mokelumne River over a longer period of time would be about 800 000 acre-ft. Therefore, undoubtedly, the period of record includes

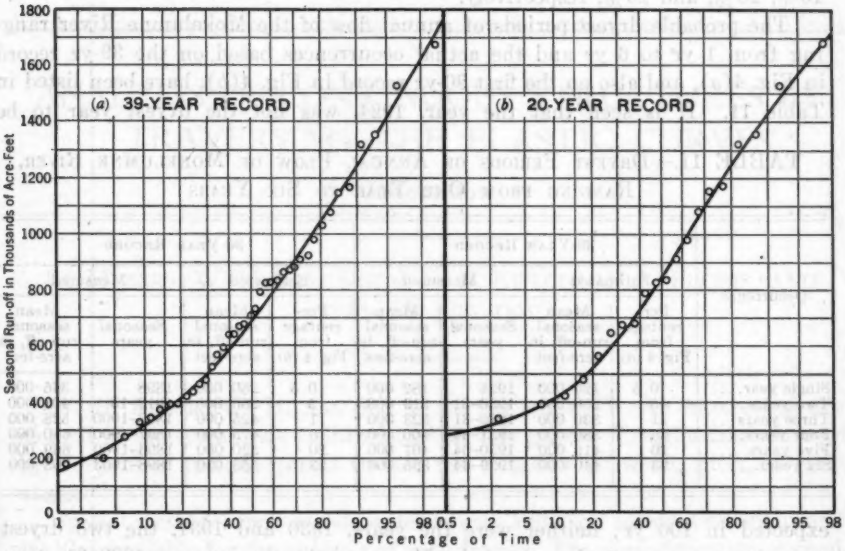

FIG. 4.—MOKELUMNE RIVER AT CLEMENTS, CALIF.: DURATION CURVE OF RUN-OFF FOR THE SEASONAL YEARS ENDING SEPTEMBER 30, 1934.

a preponderance of run-off below normal. From Fig. 4(*a*), as drawn, it is probable that the minimum run-off in a 100-yr period can be determined with a greater degree of accuracy than the maximum run-off.

This is perhaps more evident by comparing the curve in Fig. 4(*a*) with a similar curve prepared from the first 20 yr of record as depicted in Fig. 4(*b*). The latter is the curve of natural run-off from 1895–96 to the season, 1914–15, measured and estimated. From Fig. 4(*a*) it would appear that once in 100 yr a seasonal run-off of less than 150 000 acre-ft could be expected. From Fig. 4(*b*), the minimum in the same period of time is 300 000 acre-ft. There would have been no justification on the basis of the 20-yr record of estimating a minimum of 150 000 acre-ft.

In connection with reservoir studies it is often probable that two, three, or four dry years in succession will create a more serious draft on storage than a single dry year. The average probable minimum yearly flows during such dry periods can be determined readily from the duration curve. The location of the proper percentage can be determined mathematically according to the laws of chance. The probability of several successive occurrences in a period of time is the *n*th root of the period selected. Thus, the two dryest successive years in a 100-yr period would be less than $\sqrt{0.01}$, or 0.10; that is, two years in succession would occur in a 100-yr period of lesser volume than the run-off, which would normally occur once in 10 yr. The average of two successive

years can be assumed to be equal to the median point or to the run-off which would occur 5% of the time. Similarly, the average three, four, five, and six successive dry years in a 100-yr period can be found at the ordinates, 11%, 16%, 20%, and 23%, respectively.

The probable dryest periods of annual flow of the Mokelumne River ranging from 1 yr to 6 yr and the actual occurrences based on the 39-yr record in Fig. 4(*a*), and also on the first 20-yr record in Fig. 4(*b*), have been listed in Table 11. It is seen that the year, 1924, was not the dryest year to be

TABLE 11.—DRYEST PERIODS OF ANNUAL FLOW OF MOKELUMNE RIVER, RANGING FROM ONE YEAR TO SIX YEARS

Occurrence	39-YEAR RECORD				20-YEAR RECORD			
	Estimated		Measured		Estimated		Measured	
	Percentage from Fig. 4 (*a*)	Mean seasonal run-off, in acre-feet	Seasonal years	Mean seasonal run-off, in acre-feet	Percentage from Fig. 4 (*b*)	Mean seasonal run-off, in acre-feet	Seasonal years	Mean seasonal run-off, in acre-feet
Single year.....	0.5	120 000	1924	182 000	0.5	280 000	1898	346 000
Two years.....	5	250 000	1930–31	319 000	5	360 000	1912–13	408 000
Three years.....	11	330 000	1929–31	323 000	11	425 000	1898–1900	528 000
Four years.....	16	380 000	1931–34	400 000	16	475 000	1897–1900	640 000
Five years.....	20	415 000	1930–34	407 000	20	520 000	1896–1900	669 000
Six years.....	23	440 000	1929–34	395 000	23	555 000	1898–1903	708 000

expected in 100 yr; neither were the years, 1930 and 1931, the two dryest years in succession to be expected. However, the three years, 1929–31, were about equal to the dryest period of this length to be expected, and the years, 1929 to 1934, were appreciably dryer than the normal 6-yr period to be expected in 100 yr.

If the first 20-yr record is considered alone, it will be noted that the estimated occurrences for all periods from 1 yr to 6 yr are considerable lower than the actual record. This could be accepted as an indication that the dryest cycle of years to be expected had not yet been experienced. This diagnosis was amply borne out by the succeeding nineteen years of record.

In the case of the full 39-yr record, the close agreement between the measured and the estimated occurrence for periods from 3 to 6 yr would indicate that, at present, the minimum year can be determined with a reasonable degree of accuracy. In fact, if a wet cycle of years should follow, the record for 60 yr might indicate a revision of the 100-yr minimum upward.

A method of determining the probable accuracy of the maximum and minimum end-points is a necessary adjunct to the use of probability methods to statistics.

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DISCUSSIONS

ANALYSIS OF CONTINUOUS STRUCTURES BY TRAVERSING THE ELASTIC CURVES

Discussion

BY MESSRS. W. H. KIRKBRIDE, R. B. KETCHUM, A. FLORIS, AND IVAN M. NELIDOV

W. H. KIRKBRIDE,[16] M. AM. SOC. C. E. (by letter).[16a]—An interesting new method of stress analysis is introduced in this paper. The illustrations used by the author include portal frames the members of which meet at right angles.[16b]

Special methods, such as "slope deflection" and "end moment distribution," are also available for the analysis of such frames. The slope deflection method, while in quite general use for rectangular frames, is not accepted as suitable for the analysis of frames having sloping tops. The end moment distribution method is also particularly adapted to rectangular frames. It does not permit of the derivation of a general formula in terms of symbols, which can be compiled into a handbook, and becomes less convenient when applied to frames the joints of which are deflected from their original position, due to the action of the loads.

It will be interesting to know whether the traverse method is applicable to frames with sloping tops.

R. B. KETCHUM,[17] M. AM. SOC. C. E. (by letter)[17a].—The various methods of analyzing continuous structures have involved the work of distortion, or the geometry of distortion, and Mr. Stewart's paper is a distinct contribution to the geometrical method of attack.

The paper seems to have possibilities of expansion in a wide range of problems and would be a valuable check in graphical methods of drawing elastic curves and studying complicated loading conditions that have many

NOTE.—The paper by Ralph W. Stewart, M. Am. Soc. C. E., was published in October, 1934, *Proceedings*. Discussion on the paper has appeared in *Proceedings*, as follows: December, 1934, by Messrs. Garrett B. Drummond, Austin H. Reeves, E. G. Paulet, Adolphus Mitchell, and David M. Wilson.

[16] Chf. Engr., S. P. Co., San Francisco, Calif.

[16a] Received by the Secretary December 12, 1934.

[16b] Correct the moments of inertia in Fig. 4, as follows: The left column should be I; the right column, I_2; and the beams, I_3.

[17] Dean, School of Mines and Eng., Univ. of Utah, Salt Lake City, Utah.

[17a] Received by the Secretary December 13, 1934.

unknowns, for which "cut-and-try" methods are used. The writer would be interested to know whether the author has applied this method to frames with sloping tops.

A. FLORIS,[18] Esq. (by letter).[18a]—The methods for the analysis of statically indeterminate structures can be grouped into two main categories: Those in which forces are used as the redundants and those in which moments are utilized for the same purpose. They are termed the "force" and "deformation" methods, respectively.

The author's analysis is pre-eminently a deformation method with the moments at the corners and over the supports as the redundants. By replacing the deformed structure by the traverse (that is, by a set of straight lines), its flexure can be more readily visualized. The intersections of these lines determine the angles corresponding to the bending moment areas of simple beams and cantilevers divided by the stiffness factor, $E\,I$. This interpretation of the elastic curve has the additional advantage that many of the aforementioned angles can be determined by inspection. The seemingly great number of unknown angles is thus reduced considerably, and the solution of complicated problems is rendered more simple. The merit of the present analysis is based on this fact. As a further development of the traverse method, Mohr's theorem pertaining to rotations and deflections of loaded beams is utilized.

The traverse method is theoretically interesting and practically valuable. It is hoped, therefore, that the method will be used by engineers engaged in the design of statically indetermine structures.

IVAN M. NELIDOV,[19] Assoc. M. Am. Soc. C. E. (by letter).[19a]—This is a method by which moments in a continuous structure can be found with the aid of a traverse, the deflection angles of which correspond to $\frac{1}{E\,I}$ times the moment areas of the moments involved.

The first consideration in analyzing a structure is the balance between the unknown factors and the available equations. If a structure consists of m members with j interior joints and s supports, the total number of unknowns is: $n = 5\,m$. The number of available equations is:

1.—m-equations of geometry of the type,

$$\Sigma\,(\theta + \Delta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

2.—m-equations of geometry of the type,

$$\Sigma\,d = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

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[18] Civ. Engr., Los Angeles, Calif.

[18a] Received by the Secretary January 25, 1935.

[19] Senior Engr. of Hydr. Structure Design, State Dept. of Public Works, Sacramento, Calif.

[19a] Received by the Secretary January 19, 1935.

3.—$\Sigma\ (x - 1)$ j-conditions of rigidity in the joints, of the type,

$$\theta_k = \theta_{k+1} \quad \ldots\ldots(24)$$

(in which, x is the number of members meeting at the joint).

4.—j-conditions of equilibrium in the joints, of the type,

$$\Sigma M_k = 0 \quad \ldots\ldots(25)$$

5.—s-conditions at the supports, of the type,

$$\theta_s = a \quad \ldots\ldots(26)$$

or,

$$M_s = b \quad \ldots\ldots(27)$$

6.—A certain number of conditions of rotation at the joints, of the type,

$$\theta_k = c \quad \ldots\ldots(28)$$

7.—A certain number of conditions of deflection at the joints, of the type,

$$d_k = d \quad \ldots\ldots(29)$$

8.—A certain number of conditions resulting from the symmetry of load and the structure.

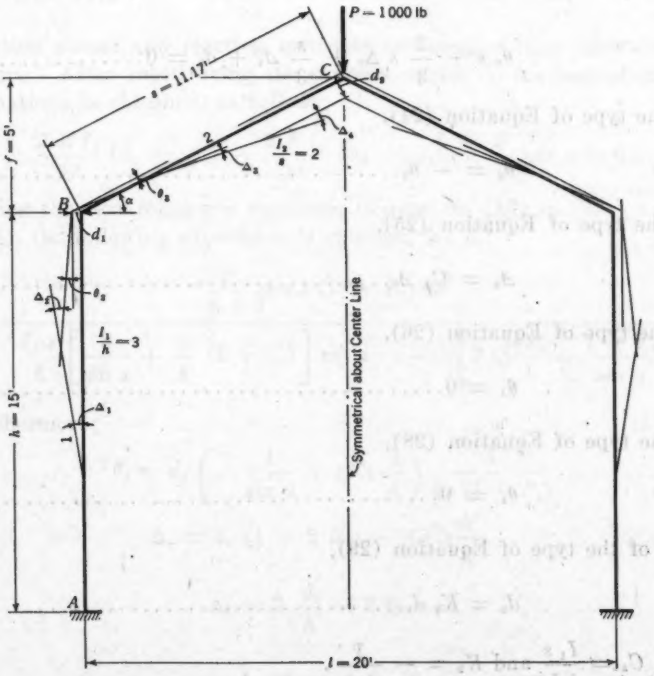

FIG. 9.

Conditions 1 to 8 reduce the original number of the unknowns, but since there are more conditions ordinarily than equations, the deficient number of equations should be supplied from the conditions of the equilibrium of the external and the internal forces.

As a simple illustration of the application of Mr. Stewart's method and of Conditions 1 to 8, the following example is proposed: Find the moments in the frame shown in Fig. 9, which is symmetrical about its center line and symmetrically loaded by a concentrated load at the apex. It will be seen that $m = 4$, $j = 3$, and $s = 2$. The total number of unknowns is $n = 5\ m = 20$, which is reduced, due to symmetry of the frame and loading, to 10. The available formulas are: Two of the type of Equation (22),

$$\theta_1 + \Delta_1 + \Delta_2 + \theta_2 = 0 \quad \ldots\ldots (30)$$

and,

$$\theta_3 + \Delta_3 + \Delta_4 + \theta_4 = 0 \quad \ldots\ldots (31)$$

two of the type of Equation (23),

$$\theta_1 h + \frac{2}{3} h \Delta_1 + \frac{h}{3} \Delta_2 + d_1 = 0 \quad \ldots\ldots (32)$$

and,

$$\theta_3 s + \frac{2}{3} s \Delta_3 + \frac{s}{3} \Delta_4 + d_2 = 0 \quad \ldots\ldots (33)$$

one of the type of Equation (24),

$$\theta_3 = -\theta_2 \quad \ldots\ldots (34)$$

one of the type of Equation (25),

$$\Delta_3 = C_3 \Delta_2 \quad \ldots\ldots (35)$$

one of the type of Equation (26),

$$\theta_1 = 0 \quad \ldots\ldots (36)$$

one of the type of Equation (28),

$$\theta_4 = 0 \quad \ldots\ldots (37)$$

and, one of the type of Equation (29),

$$d_2 = K_2 d_1 \quad \ldots\ldots (38)$$

in which, $C_3 = \dfrac{I_1 s}{h I_2}$ and $K_2 = -\dfrac{1}{\sin \alpha}$.

One more equation should be added to Equations (30) to (38) and this results from the condition of equilibrium of the joint, B, as follows:

Since $\Sigma X = 0$:

$$- T_2 \cos \alpha + V''_1 + V'_2 \sin \alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (39)$$

and $\Sigma Y = 0$:

$$T_2 \sin \alpha + V'_2 \cos \alpha - \frac{P}{2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (40)$$

In Equations (39) and (40) T_2 is the normal stress in Member 2; V' is the shear at the left support, and V'' is the shear at the right support, with the subscript corresponding to the number of the member. After T_2 has been eliminated the tenth equation is obtained:

$$V''_1 \sin \alpha + V'_2 - \frac{P}{2} \cos \alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

In order to be able to use Equation (41), the shears, V, should be expressed in terms of Δ:

$$V''_1 = \frac{M'_1 + M''_1}{h} = \frac{2 E I_1}{h^2} (\Delta_1 - \Delta_2) \quad \ldots\ldots\ldots\ldots (42)$$

and,

$$V'_2 = - \frac{M'_2 + M''_2}{s} = - \frac{2 E I_2}{s^2} (\Delta_3 - \Delta_4) \quad \ldots\ldots\ldots\ldots (43)$$

Reaction shears and reaction moments are used, with a clockwise action as positive. After substituting the expressions for V the last of the necessary equations is obtained, as follows:

$$\frac{2 E I_1}{h^2} (\Delta_1 - \Delta_2) - \frac{2 E I_2}{s^2} (\Delta_3 - \Delta_4) - \frac{P}{2} \cos \alpha = 0 \quad \ldots\ldots\ldots\ldots (44)$$

Solving the remaining five equations (Equations (30) to (33)) and Equation (44), the following expression is obtained for d_1:

$$d_1 = \frac{\frac{P}{6 E} \cos \alpha (1 + C_3) h s^2}{- \frac{I_1 s}{h} \left[\frac{3}{\sin \alpha} + \frac{s}{h} (4 + C_3) \right] \sin \alpha - \frac{I_2 h}{s} \left[3 C_3 \frac{s}{h} + \frac{1 + C_3}{\sin \alpha} \right]} \quad \ldots\ldots (45)$$

Furthermore:

$$\theta_2 = d_1 \left(- \frac{1}{\sin \alpha} + s C_3 \frac{1}{h} \right) \frac{1}{\frac{2}{3} s (1 + C_3)} \quad \ldots\ldots\ldots\ldots (46)$$

$$\Delta_4 = \theta_2 (1 + 2 C_3) - 3 C_3 \frac{d_1}{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots (47)$$

$$\Delta_2 = 3 \frac{d_1}{h} - 2 \theta_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48)$$

and,

$$\Delta_1 = - \Delta_2 - \theta_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$

Substituting numerical values in Equations (45) to (49): $d_1 = -\frac{1}{E}\,954.199$; $d_2 = \frac{1}{E}\,2132.289$; $\theta_2 = \frac{1}{E}\,57.252$; $\theta_3 = -\frac{1}{E}\,57.252$; $\Delta_1 = \frac{1}{E}\,248.092$; $\Delta_2 = -\frac{1}{E}\,305.343$; $\Delta_3 = -\frac{1}{E}\,458.015$; and, $\Delta_4 = \frac{1}{E}\,515.267$.

The corresponding reaction moments will be:

$$M'_1 = \frac{2EI_1\Delta_1}{h} = \frac{2E\,3\times 1}{E}\,248.092 = 1\,488.550 \text{ ft-lb} = M_a$$

$$M''_1 = -\frac{2EI_1\Delta_2}{h} = -\frac{2E\,3\times 1}{E}\,(-305.343) = 1\,832.061 \text{ ft-lb} = M_b$$

$$M'_2 = \frac{2EI_2\Delta_3}{s} = \frac{2E\,2\times 1}{E}\,(-458.015) = -1\,832.061 \text{ ft-lb} = M_b$$

and,

$$M''_2 = -\frac{2EI_2\Delta_4}{s} = -\frac{2E\,2\times 1}{E}\,515.267 = -2\,061.069 \text{ ft-lb} = M_c$$

The foregoing example demonstrates that Mr. Stewart's method has unquestionable advantages when the supports are prevented from being displaced laterally or when these displacements are symmetrical. On the other hand, when these conditions are not fulfilled, the method becomes as complicated as the slope-deflection method, if not more so.

The writer also wishes to note the importance of revealing physical manifestation of any evolved theory, in order to simplify its understanding. The author did this in a clear and concise manner in another paper[20] when the demonstrated the similarity between the elastic line and a railway transition spiral; and also when he mentioned that the concept of the method is the product of a slope and a distance, rather than the moment of a quantity times its lever arm. These explanations appeared to the writer as being very helpful in his attempt to grasp the method.

[20] "Improved Method of Finding Beam Deflections", by Ralph W. Stewart, M. Am. Soc. C. E., *Civil Engineering*, February, 1934.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

RATIONAL DESIGN OF STEEL COLUMNS

Discussion

BY MESSRS. WILLIAM R. OSGOOD, ALFRED S. NILES, J. F. BAKER, AND K. L. DEBLOIS

WILLIAM R. OSGOOD,[17] Assoc. M. AM. SOC. C. E. (by letter).[17a]—Since the design of compression members which would fail by buckling, if they failed, is admittedly the element in structural design least satisfactory to handle, any paper that again brings to the attention of engineers the intrinsic difficulties involved is worth while.

Mr. Young does well to call attention to the collection of column curves in the A. I. S. C. Manual,[3] for hardly a better exhibit exists showing the (unavoidable) uncertainty of engineers concerning the strength of columns used in ordinary strucutres. The very existence of such a large "factor of uncertainty" among qualified engineers calls for the assumption of a correspondingly large factor of safety.

It is well known that the strength of medium-long columns is affected markedly by small eccentricities or initial curvatures. Mr. Young is concerned chiefly with such accidental departures from ideal conditions, and he proposes, as has been done time and again, to assume values for these quantities, which are quite unknown in an actual structure, and then to design on the basis of maximum fiber stress. This procedure, as has also been pointed out before, is merely a method of applying a variable factor of safety. It is not a very good method because it results in excessive safety for short columns, as Fig. 8 indicates, and it makes the H-section appear stronger than the rectangular section, as Figs. 9 and 10 indicate, whereas the H-section is actually the weakest of all symmetrical cross-sections. The inherently safest columns so far as the effects of accidental departures from ideal conditions are concerned are, on the one hand, very short columns in which, if failure occurs, the material itself fails, and buckling plays no part, and, on the other hand, very long columns which, if they fail, fail by pure buckling, and in which high strength of the material is of no consequence.

NOTE.—The paper by D. H. Young, Jun. Am. Soc. C. E., was published in December, 1934, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[17] Materials Testing Engr., National Bureau of Standards, Washington, D. C.

[17a] Received by the Secretary December 29, 1934.

[3] "Steel Construction" Manual of the Am. Inst. of Steel Construction, First Edition, 1933, p. 176.

It cannot be emphasized too strongly that the strengths of columns, except very short ones, do not depend on any one particular stress, such as the yield point, or any other. The strength depends on the compressive stress-strain diagram as the double-modulus theory of column action shows.[18]

Mr. Young speaks of the proportional limit in the same breath with yield point and ultimate strength, yet while the last two stresses are readily, and fairly definitely, determinable, the proportional limit may be obtained as any stress from almost zero to nearly the yield point, depending on who does it and how he does it. For practical purposes in the design of columns the highest stress for which Euler's formula applies may be considered as the proportional limit. This stress will be well above a proportional limit carefully determined from a difference curve but also well below the yield point of most structural steels. It is entirely unjustifiable, therefore, to assume the ideal stress-strain diagram of Fig. 2 in explaining the behavior of columns, except possibly at large values of the eccentricity ratio. The whole problem of column action is indeed simplified on the assumption of such a diagram; in fact, there is little left, for it is precisely in the range of stresses between the highest stress for which Euler's formula is applicable and the yield point that the action of columns is most erratic.

ALFRED S. NILES,[19] ASSOC. M. AM. SOC. C. E. (by letter).[19a]—The formulas developed in Section 4 to determine the maximum stress in cases of eccentric loading are not as convenient for practical use as the equivalent ones that can be derived from the same basic theory. In the case of a member with axial compression and end moments, but no transverse load, the distance, x_m, from the left end of the span to the point of maximum moment is given by the formula[20] (using the author's notation):

$$\tan q\, x_m = \frac{e_1 - e_o \cos \phi}{e_o \sin \phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

Equation (54) gives the location at which the slope of the moment curve is zero. If $e_o > e_1$ and tan $q\, x$ is negative, this point will be off the span, and the true maximum moment will be that at the left support. If tan $q\, x$ is positive, the maximum moment will be on the span, and its magnitude will be:

$$M_m = \frac{P\, e_o}{\cos q\, x_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (55)$$

In practice, it is quite as easy to apply the criterion of the sign of tan $q\, x_m$ as to use Equation (18) or Equation (24). If it is then found that the maximum moment is in the span, it is easier to find the secant or cosine of an angle, the tangent of which is known, than to compute the quantity, ψ, which appears in Equations (21) and (25). If this line of attack is followed, Equation (25) would be superseded by,

$$M_m = M_o \sec q\, x_m = \frac{M_o}{\cos q\, x_m} \quad \ldots\ldots\ldots\ldots\ldots (56)$$

[18] *Civil Engineering*, March, 1935.

[19] Prof. of Aeronautic Eng., Stanford Univ., Stanford University, Calif.

[19a] Received by the Secretary January 28, 1935.

[20] "Airplane Structures," by Niles and Newell, John Wiley & Sons, 1929, p. 201.

and Equation (22) by,

$$S_y = \frac{f_y}{1 + \frac{e_o}{k \cos q x_m}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (57)$$

The forms with the cosine are usually more convenient than those with the secant because tables of the former function are more common. If the cosine, $q\ x_m$, is very small, however, it is better to compute the secant directly from the relation, $\sec^2 x = \tan^2 x + 1$.

Just as the computation for maximum moment can be simplified by making the determination of the location of that moment a step in the computation, so can the computation of maximum shear. The expression for the total shear at any point distant x from the left end of an eccentrically loaded strut can be written,[20]

$$V = P \left[\frac{e_1 - e_o \cos \phi}{\sin \phi} q \cos q x - e_o q \sin q x \right] \quad \ldots\ldots\ldots (58)$$

The slope of the shear curve will be zero when,

$$\cot q x_s = - \frac{e_1 - e_o \cos \phi}{e_o \sin \phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (59)$$

When $\alpha = \frac{e_1}{e_o}$ is positive, Equation (59) will indicate a location that is off the span, and the maximum shear will be at the right-hand end of the span (assuming $e_1 < e_o$). Its magnitude divided by the area of the section will be,

$$v = \frac{V}{A} = s\ e_o\ q\ (1 - \alpha \cos \phi) \csc \phi = s\ e_o\ q\ \beta \csc \phi \ldots\ldots (60)$$

When α is negative, the shear stress at the point of inflection is given by,

$$v = s\ q\ e_o \csc q\ x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (61)$$

Equations (60 and (61) can be proved to be identical with Equations (43) and (44), and are much easier to use.

Continuing along the same lines, Equations (45) to (49), inclusive, can be replaced by,

$$v_y = k\ q\ (f_y - s_y)\ F \ldots\ldots\ldots\ldots\ldots\ldots\ldots (62)$$

in which, F is a numerical factor, as follows: For $\alpha > 0$ and M_{max} at $x = 0$:

$$F = (1 - \alpha \cos \phi) \csc \phi = \beta \csc \phi \ldots\ldots\ldots\ldots (63)$$

for $\alpha > 0$ and M_{max} at $x = x_m$:

$$F = (1 - \alpha \cos \phi) \csc \phi \cos q\ x_m = \beta \csc \phi \cos q\ x_m \ldots\ldots (64)$$

for $\alpha <$ and M_{max} at $x = 0$:

$$F = \csc q\ x_s = \sec q\ x_m \ldots\ldots\ldots\ldots\ldots\ldots\ldots (65)$$

and, for $\alpha < 0$ and M_{max} at $x = x_m$:

$$F = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (66)$$

It is to be noted that Equation (63) is a special case of Equation (64) for $x_m = 0$. The criteria for absolute maximum shear represented by Equations (50) and (53) remain unchanged, but Equations (51) and (52) may be replaced by,

$$\frac{f_y - 3s}{f_y - s} = \phi \left[\cot \phi - \frac{\alpha \sin \phi}{\beta} \right] + q x_m \tan q x_m \quad \ldots\ldots\ldots (67)$$

and,

$$\frac{f_y - 3s}{f_y - s} = - q x_m \tan q x_m \quad \ldots\ldots\ldots\ldots\ldots\ldots (68)$$

Equations (54) to (68) would be more convenient for plotting curves such as those of Figs. 12, 16, and 18, than the corresponding equations in the author's paper. The first step in the analysis of an eccentrically loaded strut would be the determination of the location of the point of zero slope of the moment curve from Equation (54). The distance from this point to the point of inflection, multiplied by q, will be $\frac{\pi}{2}$. It can then be seen which of the four cases is represented. Some saving of time can probably be obtained by taking advantage of the relations between Equations (63) and (64) and Equations (50), (67), and (68).

Similar formulas for locating the point of maximum moment, etc., can easily be derived for cases of axially loaded beams. Formulas for the total shear, bending moment, slope, and deflection for single span, and continuous beams of this type, with various types of transverse load, based on the work of Müller-Breslau, have been published by the writer.[21]

In the text following Equation (22), the author recommends the addition of assumed eccentricities to allow for imperfection of manufacture. When the known eccentricity, e_1, is of the same sign as e_0, it is obvious that assumed eccentricities should be added. The author might have brought out more clearly that when e_1 and e_0 are of opposite sign the assumed eccentricities should be added algebraically rather than arithmetically. Only in this way will the assumed value of the distance from the left end to the point of inflection be increased, as is necessary if the correction is to be on the conservative side.

Although some of the author's formulas could be simplified, his basic ideas and his developments from them are sound, making his paper a valuable contribution to the literature of column theory.

J. F. BAKER,[22] ASSOC. M. AM. SOC. C. E. (by letter).[22a]—Many of the methods used to-day for the design of compression members are so unsatisfactory that any paper advocating a rational approach should be carefully studied by all structural engineers. Mr. Young's attack of the problem is most sound and he may be interested to know that certain of his suggestions have already been put into practice.

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[21] "Airplane Structures," by Niles and Newell, 1929, pp. 185–237.

[22] Prof. of Civ. Eng., Univ. of Bristol; and Technical Officer, Steel Structures Research Comm., London, England.

[22a] Received by the Secretary January 30, 1935.

His proposal to take a definite initial curvature of the axis of the member to represent all imperfections, find the load that first produces the yield stress, and then to divide this load by a constant factor to arrive at the working load was used in deriving the strut formula contained in the Steel Structures Research Committee's Recommendations for a Code of Practice for the Use of Structural Steel in Buildings, published in 1931.[23] These recommendations were adopted by the London County Council and also by the British Standards Institution, the formula being included in the London County Council's revised Code of Practice, for the design of buildings erected in London, which came into force on February 16, 1932, and in the British Standard Specification (No. 449) for the Use of Structural Steel in Buildings published in April, 1932. Thus, for the last three years almost all stanchions of steel framed buildings erected in Great Britain and many other parts of the world have been designed on this rational basis. The formula,

$$n f = \frac{f_y + (\eta + 1)\,\epsilon_e}{2} - \sqrt{\frac{f_y + (\eta + 1)\,\epsilon_e}{2} - f_y\,\epsilon_e} \quad \ldots\ldots\ldots (69)$$

in which (following the notation of the paper), f = working stress, in tons per square inch of gross section; n = factor of safety = 2.36; f_y = yield-point stress = 18 tons per sq in.; ϵ_e = the Eulerian value, $\frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$, in tons per square inch; and $\eta = 0.003\,\frac{L}{r}$. Equation (69) is merely a convenient form of the Perry formula[24] for the maximum stress in an initially curved, pin-ended, compression member which has been derived by Mr. Young as Equation (13). The measure of the initial curvature, η, was adopted as a result of Professor Andrew Robertson's study[25] of a large number of tests on struts, whereas the factor, 2.36, was chosen from a consideration of existing strut formulas.

The Recommendations for a Code of Practice were published shortly after the Committee's inception and were only tentative. As the work involved was very lengthy, it was found impossible at the time to include a rational rule for the design of stanchions subjected to end bending moments. It would have been useless to provide the designer of a steel framed building with a strut formula that necessitated the accurate determination of the end moments since in the state of knowledge then existing it was impossible to estimate, with any accuracy, the moments transmitted from the beams through the steelwork connections. As Mr. Young states, the labor involved is considerable, but in the four years since 1931 a number of investigators working for the Steel Structures Research Committee have concentrated on

[23] First Rept., Steel Structures Research Committee, Lond., H. M. Stationery Office. London, England, 1931.

[24] "On Struts," by Ayrton and Perry, *The Engineer*, Lond., Vol. LXI, 1886.

[25] "The Strength of Struts," by A. Robertson, *Selected Engineering Paper No. 28*, Inst. C. E., Lond., 1925.

the problem and their work has been under consideration for some months. While it is not possible to state exactly what the Committee's final findings will be, it is probable that design curves will be produced having the same basis as set forth by Mr. Young and applicable to stanchion lengths subjected to end moments.

Some indication of the argument which is being followed can be obtained from a paper[26] in the Committee's interim report. In that paper a method is given of dealing with an initially curved strut subjected to unequal end moments, which covers much the same ground as that of Section 4 in Mr. Young's paper. The method is interesting in that it uses a graphical construction due to Mr. H. B. Howard.[27]

For example, in Fig. 19, draw Lines OA and OB through the pole, O, in such a manner as to make equal angles, $\theta = \frac{L}{2}\sqrt{\frac{P}{EI}}$, with the center line, OZ. These lines are made to represent M_a and M_b to some definite

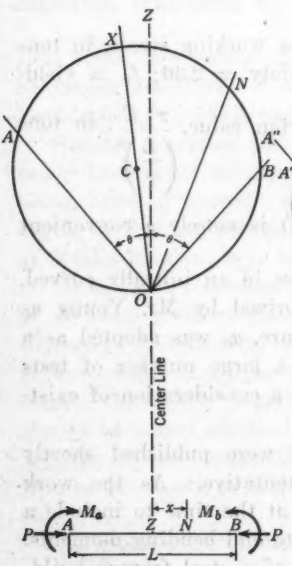

FIG. 19.

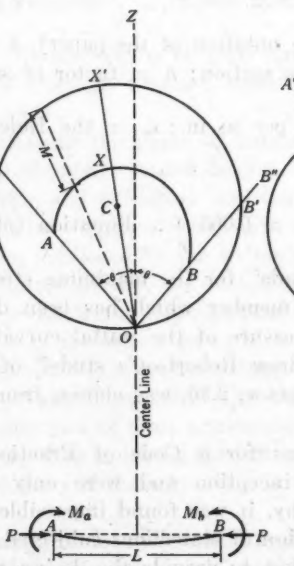

FIG. 20.

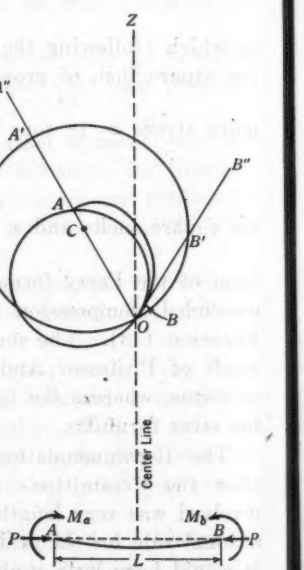

FIG. 21

scale, being measured upward from O for positive moments. A circle, $BXAO$ (Fig. 19), is constructed through Points B, O, and A. The bending moment at Point N on the beam, distant x from the center line, is then

[26] "A Note on the Effective Length of a Pillar Forming Part of a Continuous Member in a Building Frame," by J. F. Baker, Assoc. M. Am. Soc. C. E., Second Rept. of the Steel Structures Research Committee, Lond., H. M. Stationery Office, 1934.

[27] "The Graphical and Analytical Determination of Stresses in Single Span and Continuous Beams Under End Compression and Lateral Load with Variations in Shear, Distributed Load, and Moment of Inertia," by H. B. Howard, Reports and *Memoranda No. 1233*, Aeronautical Research Committee, Lond., H. M. Stationery Office, 1929.

represented by the vector, ON, which is drawn so that $< NOZ = x\sqrt{\frac{P}{EI}}$. The maximum bending moment in the beam is given by the length of the diameter, OX.

Before this construction can be used for a practical column some method of allowing for the initial curvature representing the imperfections of the member must be found. It is most usual to assume the initial shape to be that of a sine curve; there is nothing axiomatic, however, about that curve, and the stresses in the column would not be seriously affected by a slight change in its initial shape. Table 1 gives a comparison between the sine curve,

$$y = 0.003\,\frac{L\,r}{a}\,\sin\frac{\pi\,x}{L} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (70)$$

which is the initial curvature assumed in the Committee's formula (Equation (69)) and the shape assumed by an originally straight beam when subjected to a uniform moment, M, such that,

$$M = 0.024\,\frac{r\,E\,I}{a\,L} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (71)$$

in which, a is the half width of the column and I is its moment of inertia.

TABLE 1.—Comparison of Shapes for Initial Curvature of Column

x		$y\frac{a}{L\,r}$	
		Sine curve, Equation (70)	Beam under uniform moment, Equation (71)
0	L	0	0
$\frac{L}{8}$	$\frac{7\,L}{8}$	0.00115	0.00131
$\frac{L}{4}$	$\frac{3\,L}{4}$	0.00212	0.00225
$\frac{3\,L}{8}$	$\frac{5\,L}{8}$	0.00280	0.00282
$\frac{L}{2}$	.....	0.00300	0.00300

It is clear that no serious error will arise if this latter shape is taken to represent the initial curvature of the column. Therefore, a graph may be constructed for the approximate bending moment at any point of an initially curved column subjected to end moments, M_a and M_b, as follows: Lines OA'' and OB'', Fig. 20, are drawn so that they make equal angles, $\theta\left(=\frac{L}{2}\sqrt{\frac{P}{EI}}\right)$ with the center line, OZ. Along these two lines, OA' and OB' are measured in such a manner that $OA' = M + M_a$ and $OB' = M + M_b$, in which, $M\left(=0.024\,\frac{r\,EI}{a\,L}\right)$ is the uniform moment necessary to produce the initial curvature in the column. The circle, $A'\,X'\,B'\,O$, in Fig. 20 gives the bending moment in an initially straight column subjected to end moments,

$(M + M_a)$ and $(M + M_b)$; the bending moment required at any point of the initially curved column can be read from the figure, AXB, obtained from the circle, $A'\ X'\ B'$, by subtracting the moment, M, from each vector. If Line OX' is a diameter of the circle, $A'\ X'\ B'\ O$, the maximum bending moment in the initially curved column will be given by the length of the intercept, OX.

It will be seen from Figs. 20 and 21 that when the center, C, of the circle, $A'OB'$, lies on Line OA'', or on Line OB'', or anywhere on the opposite sides of those lines from the position shown in Fig. 20, the maximum bending moment will be found at one end or other of the column. When this occurs the end moments will have completely masked the "strut action" and there seems to be no reason why the permissible stress in the column should be cut down below the value, 8 tons per sq in., allowed in a beam. The critical values of the end moments which make C lie on Line OA'' or on Line OB'' are, therefore, of considerable importance.

The critical values are given by,

$$\frac{M_b + M}{M_a + M} = \cos 2\,\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (72)$$

in which, M_a and M_b are the applied end moments and,

$$M = 0.024\,\frac{r\,E\,I}{a\,L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (73)$$

The maximum bending moment in a column length, therefore, will occur at the end where the greatest end moment is applied when such moments have values defined by Equation (72); or when the greater end moment is M_a and the smaller is less than M_b; or, when the smaller end moment is M_b and the greater exceeds M_a.

In certain lengths of continuous columns a simple relation between the end moments may be assumed. For instance, in any length of an outside stanchion, except the bottom and topmost, the end moments due to dead load or to the worst combination of live loading are nearly equal in magnitude and opposite in sign, the sign convention being that of Figs. 19, 20, and 21. The critical value of the end moment is, therefore,

$$M_c = \frac{M\,(1 - \cos 2\,\theta)}{(1 + \cos 2\,\theta)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74)$$

and it is a simple matter to calculate for any length of outside stanchion, having its axis, YY, in the plane of the outside wall and subjected to any axial load, the minimum value of the end moment that will ensure the maximum moment occurring at the end.

An examination of the measured stresses in existing steel framed buildings has shown that in many cases the maximum stress occurs at an end section of the column and, therefore, the permissible stress could be taken at the full value allowed in beams.

The foregoing analysis and that given by Mr. Young only refers to end moments which bend the member in the plane containing the initial curvature. The great majority of stanchion lengths have end moments about both principal axes, and it is essential that these should be taken into account in

a practical design method. For a member bent into single curvature by end moments about both axes, all the information required is contained in Equation (75), in which the yield stress of the material is taken as 18 tons per sq in., the factor of safety is 2.36, and the allowance for initial curvature is the same as in Equation (69),

$$(7.63 - P)\left[1 - \frac{2.36\,P}{\pi^2 E}\left(\frac{L}{rx}\right)^2\right]\left[1 - \frac{2.36\,P}{\pi^2 E}\left(\frac{L}{ry}\right)^2\right]$$
$$= f_{x\text{-}x}\left[1 + \frac{0.61\,P}{\pi^2 E}\left(\frac{L}{rx}\right)^2\right]\left[1 - \frac{2.36\,P}{\pi^2 E}\left(\frac{L}{ry}\right)^2\right]$$
$$+ f_{y\text{-}y}\left[1 + \frac{0.61\,P}{\pi^2 E}\left(\frac{L}{ry}\right)^2\right]\left[1 - \frac{2.36\,P}{\pi^2 E}\left(\frac{L}{rx}\right)^2\right]$$
$$+ 0.003\,P\left(\frac{L}{ry}\right)\left[1 - \frac{2.36\,P}{\pi^2 E}\left(\frac{L}{rx}\right)^2\right] \quad \ldots\ldots\ldots\ldots (75)$$

in which, P = axial load, in tons per square inch; $f_{x\text{-}x}$ = end bending stress about the major axis, XX, in tons per square inch; and $f_{y\text{-}y}$ = end bending stress about the minor axis, YY, in tons per square inch.

From this case, design curves can be produced with ease, but the procedure is somewhat more complicated than that usual in design offices to-day. Where this complication must be avoided it can be proved that safety in design is assured in all cases if a bending stress equal to the sum of the bending stresses, $f_{x\text{-}x} + f_{y\text{-}y}$, about both axes is assumed to be due to bending about the minor axis, YY, and the simpler design curves corresponding to that case are used. Considerable economy is sacrificed, however, by this simplification.

Mr. Young has mentioned the effect of the axial load in the member on the end moments. In steel framed buildings this effect of the axial load can be considered as arising from two causes: (*a*) The flexure brought about by loading the beams; and (*b*), the initial curvature. As far as the production of a safe design formula is concerned Cause (*a*) may be neglected in all cases; Cause (*b*) may be neglected when the column is bent by the beams in single curvature; but it must be taken into account when the bending is in double curvature. With the conservative allowance of initial curvature which must be made to represent the imperfections of the member, the effect may be considerably more than the 6% mentioned by Mr. Young for the modification of secondary end moments in trusses.

K. L. DeBlois,[28] Assoc. M. Am. Soc. C. E. (by letter).[28a]—The subject of a rational design for steel columns, of course, is not a new one. A glance through engineering literature will show the attempt to derive a formula which is theoretical and at the same time not too difficult to apply from the designer's point of view. Mr. Young's derivation follows closely that of the well-known secant formula except that he assumes that the eccentricity is represented by an equivalent curvature, whereas the secant formula is based on an initial end eccentricity.

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[28a] Received by the Secretary February 4, 1935.

The author points out a serious defect in the column formulas of the straight-line, parabolic, and Rankine types in general use to-day. This defect is that the relation between the axial load and the maximum fiber stress is non-linear as shown in Fig. 6. In other words, if f_m is 30 000 lb per sq in., s_y is not 1.5 times the value of s_y when f_m is 20 000 lb per sq in. This is due to the change in eccentricity as P varies. The secant formula, Equation (14), with its transcendental function takes this into account as does the author's Equation (12). This means that if tension members of a structure were designed for a certain loading and a unit stress of 18 000 lb per sq in., they would all become stressed to 26 000 lb per sq in. under a 50% increase in load; whereas, in a compression member, if the load were increased 50% the unit stress would be greater than 26 000 lb per sq in. This is important because many bridge design specifications require that all truss members shall be uniformly stressed under a certain increase in the live load.[29]

The term, $\frac{e}{k}$, in Equation (14) can be written $\frac{e\,c}{r^2}$ since $k = \frac{S}{A} = \frac{I}{cA} = \frac{A\,r^2}{cA} = \frac{r^2}{c}$. The formula is then in its usual form,

$$S = \frac{f_m}{1 + \frac{e\,c}{r^2} \sec\left(\frac{L}{2r}\sqrt{\frac{s}{E}}\right)} \qquad \text{.....(76)}$$

By substituting the term, $\left(1 + \frac{\left(\frac{L}{r}\right)^2}{X\left(\frac{c}{r}\right)}\right)$, for $\sec\left(\frac{L}{2r}\sqrt{\frac{s}{E}}\right)$, the resulting formula (termed the "Modified Secant Formula") follows the secant formula curve closely for values of $\frac{L}{r}$ less than 200[29]; thus:

$$s = \frac{f_m}{1 + \frac{e}{r}\left(\left(\frac{L}{r}\right)^2 \frac{1}{X} + \frac{c}{r}\right)} \qquad \text{.....(77)}$$

in which, X is a constant, its value depending on f_m. In railroad work a bridge is considered ready for renewal when f_m becomes 26 000 lb per sq in. Using this value of f_m in the secant formula (Equation (76)), the modified secant formula is,

$$s = \frac{26\,000}{1 + \frac{e}{r}\left(\left(\frac{L}{r}\right)^2 \frac{1}{6\,400} + \frac{c}{r}\right)} \qquad \text{.....(78)}$$

If the working stress, f_w, is 20 000 lb per sq in., Equations (76) and (78) would be divided by 1.3 to get the design unit stress. This gives slightly

[29] See "A Simplified Column Formula of the Secant Type," by J. B. Hunley, *Bulletin*, A.R.E.A., Vol. 29, No. 300, October, 1927, p. 197.

larger sections than by using f_m of 20 000 lb per sq in. in Equation (76), but the compression members would then be stressed consistently with the tension members when the load is increased 1.3 times. Thus, Equations (77) and (78) need not be solved by "trial and error" as in the case of the author's equations. It must be understood, however, that the non-linear relation between f_m and s is attained by varying the value of X in Equation (77) to fit the secant formula, (Equation (76), as different values are used for f_m. For design purposes where one or two values are used for f_m, Equations (77) and (78), providing a direct solution, are more desirable. However, in research and

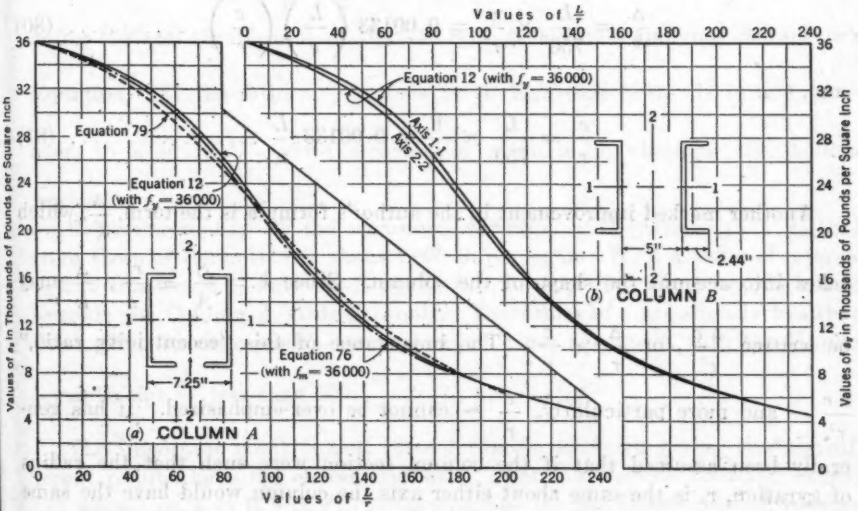

FIG. 22.—EFFECT OF $\frac{c}{r}$ -RATIO ON s_y.

investigation work where different values of f_m are encountered, Equation (12) is certainly easier to solve than the secant formula, Equation (76).

A comparison of the author's equations with the modified secant formulas for two different column sections, is shown in Fig. 22. The column character-

TABLE 2.—CHARACTERISTICS OF COLUMNS IN FIG. 22 (TWO 8-INCH CHANNELS @ 16.25 POUNDS)

Definition (1)	FIG. 22 (*a*)		FIG. 22 (*b*)	
	Axis 1–1 (2)	Axis 2–2 (3)	Axis 1–1 (4)	Axis 2–2 (5)
Section area, A, in square inches	9.52	9.52	9.52	9.52
Moment of inertia, I, in inches^4	79.6	92.7	79.6	92.7
Radius of gyration, r, in inches	2.89	3.10	2.89	3.10
Distance, c, from neutral axis to extreme fiber, in inches	4.00	3.62	4.00	4.94
Ratio, $\frac{c}{r}$	1.39	1.17	1.39	1.59
r^2	8.35	9.61	8.35	9.61
$k = \frac{r^2}{c}$	2.09	2.65	2.09	1.95

istics are shown in Table 2. Let $f_y = 36\,000$ lb per sq in.; let e and Δ both be assumed as $\frac{L}{750}$; $E = 29\,000\,000$; and X in the modified secant formula $= 2\,500$. Then Equation (77) becomes,

$$s_y = \frac{36\,000}{1 + \frac{e}{r}\left(\left(\frac{L}{r}\right)^2 \frac{1}{2\,500} + \frac{c}{r}\right)} \quad \ldots\ldots\ldots\ldots (79)$$

$$\frac{\Delta}{k} = \frac{L}{750} \times \frac{c}{r^2} = 0.00133 \left(\frac{L}{r}\right)\left(\frac{c}{r}\right) \quad \ldots\ldots\ldots\ldots (80)$$

and,

$$\frac{e}{r} = \frac{L}{750} \times \frac{1}{r} = 0.00133 \frac{L}{r} \quad \ldots\ldots\ldots\ldots (81)$$

Another marked improvement in the author's formula is the term, $\frac{\Delta}{k}$, which takes into account the shape of the column. Since $k = \frac{S}{A} = \frac{r^2}{c}$; $\frac{\Delta}{k}$ may be written $\frac{\Delta c}{r^2}$, or $\frac{\Delta}{r} \times \frac{c}{r}$. The importance of this "eccentricity ratio," $\frac{\Delta c}{r^2}$ — and more particularly, $\frac{c}{r}$ — cannot be over-emphasized. It has generally been assumed that if the column section were such that the radius of gyration, r, is the same about either axis the column would have the same strength; but this is not the case unless the ratio, $\frac{c}{r}$, is the same about both axes.[20]

The columns in Fig. 22 have the same minimum radius of gyration, r, but the ratio, $\frac{c}{r}$, for the column in Fig. 22(b) changes the limiting axis and is weaker than the column in Fig. 22(a). Present-day formulas would lead to the conclusion that the columns were equal in strength since their minimum radii of gyrations are equal; but the secant formula shows that this is not true. The limiting axis of the column in Fig. 22(b) is 2–2 even though the radius of gyration, r, is greater than about Axis 1–1. Without regard to formulas one would expect the outstanding legs of Column B to be a source of weakness. Thus, the secant type of formulas enables the designer to take into account the column shape as accurately as he now computes the radius of gyration. The most economical section occurs when $\frac{c}{r}$ is a minimum about either axis. It approaches 1.00 as a minimum. A circular section, of course, would be the

nearest approach to the ideal. The $\frac{c}{r}$ - ratio has a marked effect on s for values of $\frac{L}{r}$ from 20 to 80, a common range in design. A variation in $\frac{c}{r}$ from 1.5 to 2.5 affects the values of s as much as a variation of $\frac{L}{r}$ from 40 to 60 or from 60 to 80.

The total effect of $\frac{c}{r} \times \frac{e}{r}$ is given by the author's use of $\frac{\Delta}{k}$, but in so doing the real significance of the $\frac{c}{r}$ - ratio is lost. In applying the author's Equation (12) the ratio, $\frac{c}{r}$, appears as in Equation (80). It is best, however, to keep the $\frac{c}{r}$ - ratio separate as a guide in choosing the column section.[30]

Referring to Fig. 22 the maximum variation of Equations (77) to (81) from those of the author is about 1 000 lb per sq in. With a value of n equal to 2.25 this would be only 440 lb per sq in. On Fig. 22(b) is shown the secant formula for Column A, Axis 1–1, only. The values of s are slightly less than those given by the author's formula and in close agreement with the Hunley formulas for $\frac{L}{r}$ less than 200.

The important point is that these formulas, although giving slightly different results, have the characteristics of a rational column formula, namely (1) when e or $\Delta = 0$, $s = f$; (2) they express the non-linear relation between s and f; and (3) they take into account the shape of the column as well as the slenderness ratio.

[30] See "Modern Framed Structures," by Johnson, Bryan, and Turneaure, Pt. III, p. 69.

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DISCUSSIONS

THE SILT PROBLEM

Discussion

BY MESSRS. E. W. LANE, AND FRANK E. BONNER

E. W. LANE,[8] M. AM. SOC. C. E. (by letter).[8a]—As the development of water resources for water supply, water power, irrigation, and navigation is becoming more thorough, and the control of floods of streams in the United States is growing more urgent, the silt problem which Mr. Stevens so interestingly describes, is increasing in importance yearly; in the future, no doubt, it will be even more vital. In order to find adequate solutions for the present and prospective silt problems, a more complete development of the governing laws is necessary. At present, this science is in an undeveloped state. Although much progress has been made, the problem is very complex, due to the large number of variables involved, and much remains to be accomplished before a satisfactory science will have been developed.

Like other phases of hydraulic engineering, present knowledge of the mechanics of the solids load of streams has been a gradual growth, with its beginning many years ago. Probably the first scientific approach was made by Guglielmini near the end of the Seventeenth Century. He developed laws governing fluvial phenomena, in connection with the control of the Po River in Italy. The problem first assumed importance in the United States in connection with the controversies which arose over the control of the floods of the Mississippi River and the construction of jetties for navigation improvement at its mouth. Extensive observations of suspended load were made in the studies of Humphreys and Abbot near the middle of the Nineteenth Century, and the conclusions they reached were at variance with current opinion. These results led to a violent controversy with the late James B. Eads, F. Am. Soc. C. E., who about 1875 proposed to deepen the river mouths by jetties, and this discussion created considerable interest in solids load phenomena. Upon the formation of the Mississippi River Commission in 1879, extensive silt measurements were begun on the Mississippi and its principal tributaries, but were continued only two or three years. Except

NOTE.—The paper by J. C. Stevens, M. Am. Soc. C. E., was published in October, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: February, 1935, by Harry G. Nickle, Jun. Am. Soc. C. E.

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[8a] Received by the Secretary February 4, 1935.

for the excellent summary of the studies of engineers all over the world published by Dr. E. H. Hooker in 1896,[9] little progress was made in this country on this problem until the work of G. K. Gilbert about 1914.[10] More recently, valuable work on the filling of reservoirs has been done by T. U. Taylor,[11] M. Am. Soc. C. E. A comprehensive description of the silt problem of the Lower Colorado was published by the late Carl Ewald Grunsky, Past-President, Am. Soc. C. E.,[12] in 1929. Another excellent report[13] on the Colorado River is that by the late Samuel Fortier, M. Am. Soc. C. E., and by H. F. Blaney, Assoc. M. Am. Soc. C. E.

The last few years, however, have seen an active interest in this subject in this country due largely to the return of the Freeman Scholars from Europe, where active work has been going on for many years, and to the importance which the problem has assumed in connection with the control of the Mississippi, Missouri, and Colorado Rivers, and in a number of beach and harbor problems. The subject is now, or has recently been, under active study at the U. S. Waterways Experiment Station at Vicksburg, Miss., the National Hydraulic Laboratory, the Bureau of Reclamation Laboratory, and at the laboratories of the Universities of California, Iowa, Minnesota, and the Massachusetts Institute of Technology. The result of this activity has been a considerable increase of knowledge in the science of transportation and control of the solids load.

Because of the great range of conditions, the solids load problems of the United States have had many aspects. The principal ones were the silting of reservoirs and the control of rivers, generally in fine material, for navigation improvement. The problems in other countries have also depended on local conditions. Formerly, the studies in Europe dealt principally with the control of rivers for navigation, and were concerned with bed-load movement, generally of relatively coarse material. More recently there has been considerable interest in the bed-load movement of finer material, as an outgrowth of efforts to represent the relatively coarse river bed material by finer material in model tests. The control of mountain torrents has also assumed considerable importance in recent years, and excellent studies are now (1935) being made of the suspended load problem.

Another large field for these studies has been the irrigation systems of India. There the investigations have been confined almost entirely to designing stable channel sections and the elimination of silt from canal intakes. The first of these was the classical paper by R. G. Kennedy,[14] published in

[9] "The Suspension of Solids in Flowing Water," *Transactions*, Am. Soc. C. E. Vol. XXXVI (1896), p. 239.

[10] "Transportation of Débris by Running Water," U. S. Geological Survey, *Professional Paper No. 86*; and "Hydraulic Mining Débris in the Sierra Nevada," U. S. Geological Survey, *Professional Paper No. 105*.

[11] "Silting of Lake Austin, Texas," *Transactions*, Am. Soc. C. E., Vol. 93 (1929), p. 1681; and "Silting of Reservoirs," *Bulletin 3025*, Univ. of Texas, 1930.

[12] "Silt Transportation by the Sacramento and Colorado Rivers and by the Imperial Canal," *Transactions*, Am. Soc. C. E., Vol. 94 (1930), p. 1104.

[13] "Silt in the Colorado River and Its Relation to Irrigation," *Technical Bulletin 67*, U. S. Dept. of Agriculture.

[14] "The Prevention of Silting in Irrigation Canals," *Minutes of Proceedings*, Inst. C. E., Vol. 119, p. 281, 1895.

1895. This was followed by papers by Messrs. Lindley, Woods, Bottomley, and Griffith and, finally, the paper by Gerald Lacey.[15] The latter paper has stirred considerable interest in the subject and thorough research is in progress in several of the Provinces to determine the various factors controlling stable channel shapes.

In recent years, the silt problem of the Nile has been thoroughly studied in its relation to the silting of the irrigation ditches leading from it, but little has been published regarding the results. In the last few years considerable attention has been given to the silt in the rivers of China. In that country the problem is probably more vital than in any other. The floods of the rivers of North China are largely due to the tremendous loads of sediment brought down from their loess-covered drainage areas. The famous floods and channel changes of the Yellow River[16] are due to this cause. The silt concentrations in these rivers sometimes attain surprising values. S. Eliassen, Assoc. M. Am. Soc. C. E.,[17] reports that on the Ching River, "silt contents of 48% by weight were observed, and at one time 51%. The silt content often went above 30% and would stay above 30% for several days." He also states that in the 1933 flood the Yellow River at Sanchow Honan carried 39% by weight. O. J. Todd, M. Am. Soc. C. E., has reported 23.09% by weight on the Fen River,[18] and a maximum of 38% by weight has been reported in the Yung Ting River.[19] The only records in the United States comparable with these, as far as the writer knows, are those on the San Juan River, a tributary of the Colorado. A maximum of 40.8% by weight has been reported for this river by Howard and Love.[20]

It may seem queer that so little progress toward the development of a definite science should be made in a subject to which so much study has been given, but when one reflects on how many factors are involved, it is not surprising. The material transported by any section of a stream is the result of the interaction of two groups of factors. In the first group are those which influence the quantity and quality of the load brought down to that section of the stream. In the second group are factors which influence the capacity of the stream to transport load. A tentative list of these factors is as follows (these are not all independent and, in some cases, their effect is not definitely known):

Group 1.—Solids Brought Down to the Stream.—

(*a*) Quality: Size; specific gravity; shape; and dispersion.

(*b*) Quantity: Geology of water-shed; magnitude and distribution of rainfall; vegetal cover; cultivation and grazing; and erosion.

[15] "Stable Channels in Alluvium," *Minutes of Proceedings*, Inst. C. E., Vol. 229, p. 259.

[16] "Flood Problems in China," by the late John R. Freeman, Past-President and Hon. M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 1436.

[17] *Journal*, Assoc. of Chinese and American Engrs., September, 1935, pp. 26 and 29.

[18] Fen Ho Rept., 1933, Plate XXVI.

[19] The Abridged Rept. on the Radical Improvement Scheme for the Yung Ting Ho, 1934, p. 9.

[20] *Engineering News-Record*, Vol. 105, p. 620.

Group 2.—Capacity of Stream to Transport Solids.—

(*a*) Shape of Stream Prism: Depth; width; form; and alignment.

(*b*) Hydraulic Properties of the Stream: Slope; roughness, hydraulic radius; velocity; velocity distribution; turbulence; tractive force; temperature; and uniformity of flow.

Factors in Group 1 are subject to so many variations, not only between streams, but also at a given point of one stream, that analysis of any case in a quantitative way is impracticable. They resemble closely the factors controlling stream run-off, and, as in the case of run-off, about the only way they can be treated quantitatively is to observe them over a long period and record the results. Like records of stream run-off, solids load records are very valuable. In gathering so large a number of records from the scattered sources, Mr. Stevens has performed a valuable service. Too often a decision as to the construction of a project must be made before it is possible to collect sufficient data on the load of that stream, and recourse must be had to comparisons with loads observed on other streams. For this purpose, Mr. Stevens' data will be found to be invaluable. The present situation in regard to the solids load is much the same as that which existed in the case of stream run-off in the Nineties. In the future engineers may expect more solids load records to be collected until data will be available on all the large streams in the United States.

Although the prospect of a quantitative determination of the load brought down, derived from the various factors involved, does not seem practical, the study of the other phase, the capacity of a stream to transport material of a given quality, is making considerable progress. The relations between a number of the factors involved from the science of hydraulics is already known and the remaining ones are being subjected to close study. It is possible to study many of the these relations in the laboratory where the conditions can be controlled and the effect of varying only one factor at a time observed.

In general, the solids load is transported both in suspension and by rolling along the bottom as bed load. The intermediate stage of saltation is usually unimportant. Apparently, the laws governing these two forms of transportation are different. The most recent work on the science of bed-load movement has been along the line of the relation between load and tractive force, and a solution of this problem seems nearer than that of the suspended load. Considerable study is also being given to the determination of laws of the transportation of solids in suspension. For many years this has been a matter of controversy among engineers. In Hooker's[9] thorough review of the pertinent literature first published about 1896, he set forth the two principal theories: (1) That the solid material is held in suspension by the difference in velocities between adjacent filaments of water; and (2) that it is held in suspension by the upward movement of currents or eddies in the flowing water. Both these theories are still widely held to-day and progress will be accelerated when the false one is abandoned. The first

of these theories was developed by Dupuit[21] who rotated a glass of water containing sand grains and from the action of the sand concluded that the particles tended to move toward the fastest moving water, and, therefore, were lifted toward the surface in a stream.

The writer has never been able to accept this theory for several reasons. One is that according to it there could be no material in suspension above the point of maximum velocity in a stream, since above this point the velocity gradient would usually be downward and, therefore, there would be no force lifting the particles against the force of gravity. It is well known, however, that sediment does occur in suspension above the point of maximum velocity. Furthermore, if Dupuit's theory were valid, in a dredge pipe through which water and sand were flowing at high velocity, because of the high velocity gradients which would occur toward the center of the pipe, there would be great forces tending to move the material toward the center of the pipe and hold it there. This would leave the remainder of the pipe filled with practically clear water. No such action has been observed. A third reason is that in large, deep channels, the differences in velocity at the two sides of a small particle of silt are so minute that it seems unreasonable to believe they could generate enough force to hold the particle up against the attraction of gravity.

In order to inquire more fully into the claims of the Dupuit theory, the writer has recently experimented with sand in a revolving glass vessel, as Dupuit did, and finds as much evidence to support the turbulence theory as that of Dupuit. As a result of these experiments and the three reasons mentioned previously the writer is convinced of the fallacy of the Dupuit explanation of the suspension of solids in flowing water and confirmed in the belief that solids are carried in flowing water by the upward currents and eddies.

It is not difficult, however, to account for the persistence of the Dupuit theory through many years, in spite of the unanswerable objections which have been raised against it. Although the velocity gradient does not produce the forces causing the solid particles in suspension to move against the force of gravity, a high velocity gradient usually indicates high turbulence, and, therefore, a region of high suspending ability. The velocity gradient and suspending forces, therefore, are related but not, as Dupuit supposed, in a cause and effect relation.

The most recent studies in the suspended load problem from the theoretical standpoint in this country have been along the line of relating suspended material and turbulence. The work of Professors O'Brien and Leighly has already been mentioned in the paper. Since the paper was written, an additional paper by Professor Leighly,[22] and one by Professor von Kármán[23] have added much to an understanding of the problem.

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[21] "Etudes," par. J. Dupuit, p. 220.

[22] "Turbulence and the Transportation of Rock Débris by Streams," *The Geographical Review*, Vol. 24, No. 3, July, 1934.

[23] "Some Aspects of the Turbulence Problem," Aeronautics and Hydraulics Div., Am. Soc. M. E.

Related to the problem of turbulence and suspended load transportation is a striking phenomenon which one often sees on the surface of a canal or stream of very muddy water. This is a streak of practically clear water, surrounded by very turbid water. As this phenomenon throws some light on forces keeping particles in suspension it was closely observed by the writer. The cause of this phenomenon seems to be, as follows: When the water at the surface of a flowing stream is undisturbed for a short period the silt settles down, leaving at the surface a thin unnoticed layer of desilted water. Throughout the stream, however, there are eddy currents rising from within the water prism to the surface. In moving through the mass of water, they encounter comparatively little resistance on their upward course, the action of the force of gravity being balanced by a downward movement of an equal quantity of water in another part of the stream. When they reach the surface and attempt to rise higher, their motion is not balanced by an equivalent downward motion and their upward momentum, which is relatively small, is quickly overcome by the force of gravity. These upward currents then spread out in an umbrella shape in all directions near the surface of the stream. If two of these upward eddies come up on opposite sides of any area where there is a slight layer of clear water, as previously described, the eddies spreading out on the surface squeeze the clear water layer between them together, until it forms a narrow streak of clear water which has sufficient depth to be apparent to any observer.

In addition to being a noticeable phenomenon, this action probably is an important factor in the mechanics of the transportation of solids in suspension, as it seems to explain the reason why the maximum velocity in a stream occurs below the surface, which must be taken into account in any complete and rational treatment of the suspension of silt in streams. The mass of water in these eddies in passing up from the bottom moves through the main body of the water and retards its longitudinal motion somewhat, because having originated at the bottom, the longitudinal motion of the eddy is slower than the average of the mass. The greatest effect, however, is at the surface, where the eddies spread out and join together to form the top layer, with a lower longitudual velocity than the water immediately below. The eddies which cause the visible clear-water streaks previously mentioned are of considerable magnitude, but smaller eddies no doubt similarly contribute to the upper, slow-moving layer. There seems to be no reason why the limitation to motion imposed by the surface of the stream should not have a similar effect on the intimate internal motion which makes up the turbulences in the flowing water. As the observation on the silt transported in a stream led the writer to a better understanding of an idea of the internal mechanics of the flow of water in an open channel, it seems probable that knowledge of the internal mechanics of flowing water and its turbulence may perhaps be reached more efficiently by the study of the transportation of solids in suspension than by observations on clear water flowing in channels.

FRANK E. BONNER,[24] M. AM. SOC. C. E. (by letter).[24a]—Contributions to technical literature on the important subject of silt movement and control are increasing in desirable degree. The comprehensive paper by Mr. Stevens provides a notable addition, and the author deserves particular commendation for the large amount of effort obviously expended in the correlation of the statistical data presented by Table 6. The inclusion of the large number of determinations made by the Corps of Engineers, U. S. Army, during the past few years throughout the Mississippi Basin bridges a gap existing in previous tabulations. It is evident, however, that a number of the recorded observations cover too short a period for the determination of dependable normals, and some caution will have to be exercised in the use of the data for such purposes. The same is true in regard to part of the voluminous data abstracted from *Water Supply Paper 274* of the U. S. Geological Survey. For example, Item No. 2, Table 6, is based on measurements made at Palisade, Colo., on 160 days of an elapsed period of 399 days in 1905 and 1906. The silt load value is given as 0.49 part per thousand. However, analysis of the same data reported in *Water Supply Paper 617* gives 4 000 000 acre-ft and 1 758 700 tons (950 acre-ft at 85 lb) as the estimated annual water discharge and silt load, respectively. These data give a silt load value of 0.323 part per thousand which is less than the tabular value by about 34 per cent. Similar discrepancy between different interpretations of results is illustrated by Item No. 3, Table 6, and others. It is thus apparent that, for detailed uses, reference to the original sources of data may be desirable.

The paper refers to the reservoir silting problem with considerable emphasis on the hazards of ultimate depletion. The writer prefers to view this problem with less alarm concerning the possible misfortunes of the remote future. Somehow, these threatened disasters due to "impending exhaustion" of different elements of natural resources never seem to materialize with the serious results anticipated. Imminent shortages automatically set in motion the ingenuity of mankind toward the provision of substitutes which generally serve better than the materials displaced. Consider the lumber situation for example: Decreasing per capita consumption due to rapid increase in the use of wood substitutes has relieved the pressure on the timber resources of the United States. Experts concede that "the threat of a timber famine which loomed so large some 25 years ago has somewhat worn off."[25] Likewise, in the case of the water power resources; several decades ago extensive public concern arose over their adequacy to meet future requirements. To-day, water power development is in the discard, due to the amazing increase in the efficiency of steam-electric generation and the widespread production of cheap fuels. Past bugaboos relative to impending scarcity of fuels, farm lands, fertilizers, and other resources have gone through the same cycles. Present fashion tends toward "saving the soil from erosion" and this movement will probably have to run its course.

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[24] Cons. Engr., Piedmont, Calif.

[24a] Received by the Secretary February 18, 1935.

[25] "Place of Forestry in Land Utilization Program," by Raphel Zon, *United States Daily*, July 1, 1932.

The accumulation of silt, of course, is one of the elements to be considered in the case of every proposed reservoir. Rapid depletion of capacity may affect the economic plan of the project and, perhaps, even the design of the dam. Fortunately, however, the problem is not serious in many cases. Rapid depletion of capacity is rather well restricted to the reservoirs on the silt-laden streams of the arid Southwest and to those with capacities disproportionately small in relation to the area of their tributary water-sheds.

Elephant Butte and Boulder are frequently cited to illustrate the transitory character of reservoirs. Silt conditions at both structures were fully recognized by the builders. It is probable that no streams in the United States have been studied in regard to silt transportation more thoroughly than the Rio Grande and the Colorado River. The excessive débris-producing capacity of these water-sheds will be continuous. Herman Stabler, M. Am. Soc. C. E., has showed[26] the impracticability of arresting erosion in these arid regions by so-called "water-shed protection" methods.

Actual accumulations of silt in Elephant Butte Reservoir, since its completion in 1916, correspond closely with the forecasts based on studies made prior to its construction. Unless serious errors have been made in the economic programs of these projects, it is reasonable to assume that the capital outlays will be amortized long before sedimentation of the reservoirs seriously impairs their usefulness. When that time eventually comes, any desirable restoration of capacity will be a problem for the engineer of that distant day. If engineers attempt to provide a solution now, it would propably be found entirely wrong long before the time arrived for its application.

On the basis of present conditions, ultimate construction of new storage would doubtless be considered the proper remedy; but under the unpredictable conditions of the dim future dredging, sluicing, or some other means may be found more economic. Moreover, considering the amazing march of science in the recent past, it is conceivable that water storage requirements 50 yr or 100 yr hence may be on an entirely different basis than the present. History teaches the futility of trying to peer very far into the future.

Various estimates have been made relative to the average rate of surface leveling in the United States based on silt movement. Mr. Stevens cites an estimate published twenty-five years ago by the U. S. Geological Survey giving the annual total of suspended matter at approximately 500 000 000 tons. The accumulation of more recent data supports the writer's estimate of about 933 060 000 tons.[27] The frequently quoted figure of 3 000 000 000 tons, attributed to the Federal Soil Erosion Service, is clearly excessive.

[26] "Rise and Fall of the Public Domain," by Herman Stabler, *Civil Engineering*, September, 1932, p. 541.

[27] *Proceedings*, Am. Soc. C. E., April, 1934, Pt. 1, p. 573.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

EFFECT OF SECONDARY STRESSES UPON ULTIMATE STRENGTH

Discussion

BY MESSRS. L. E. GRINTER, L. T. EVANS, AND F. E. FAHY

L. E. GRINTER,[23] Assoc. M. Am. Soc. C. E. (by letter).[23a]—The attempt of the authors to investigate the effect of secondary stresses upon the ultimate strengths of truss members deserves sincere applause. For a period of more than half a century the question of the importance of secondary stresses has been widely argued without any successful effort to reach even a compromise agreement. Practice ranges from the entire disregard of all secondary stresses to their full inclusion at standard working stresses. The designers of a few important structures have taken the intermediate path either of attempting to eliminate secondary stresses by the introduction of opposite erection stresses, or of taking the secondary stresses into consideration at increased working stresses.

There should be no disagreement with the authors' premise that secondary stresses should be disregarded if it can be shown that they do not hasten ultimate failure of the structure. However, there are many capable designers who will not agree with this statement because it seems in conflict with the conception of a factor of safety based upon the elastic limit of the material. After all, that is certain to be the major point of disagreement. The old conception of a factor of safety based upon the ultimate strength has been shown to be fallacious and has been replaced by a factor of safety (or better, a factor of ignorance) based upon the elastic limit. Now, it is suggested that the range of plasticity that exists barely above the elastic limit should be exploited to reduce the effect of the secondary stresses and add to the ultimate theoretical capacity of the structure.

The authors' suggestion will immediately bring up the subject of proper working stresses. It has been shown repeatedly that the maximum secondary stresses in an ordinary truss approach one-third the values of the corresponding primary stresses. Hence, a member designed at 18 000 lb per sq in.

NOTE.—The paper by John I. Parcel, M. Am. Soc. C. E., and Eldred B. Murer, Jun. Am. Soc. C. E., was published in November, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1935, by Messrs. C. H. Sandberg and J. D. Gedo.

[23] Prof., Structural Eng., Agri. and Mech. Coll. of Texas, College Station, Tex.

[23a] Received by the Secretary January 14, 1935.

might actually be stressed to 24 000 lb per sq in. The range from 24 000 lb per sq in. to be the minimum elastic limit (between 30 000 and 35 000 lb per sq in.) has been considered sufficient to cover fabrication stresses, erection stresses, exceptional overload or unusual impact, and an allowance for reduction of area by corrosion. If the authors are prepared to suggest the entire neglect of secondary stresses in design, it would seem that this would permit an increase in the basic working stress of about 33 per cent. Probably most readers will be unwilling to agree to such an increase as being reasonable although they may feel that the authors have established their contention.

Although the authors point out that the "best standards of practice" require that provision for secondary stresses shall be made in design, it is well known that the great majority of designers have not had the necessary theoretical training to compute the values of the secondary stresses and, therefore, have depended upon the allowance in the working stress. Only with the introduction of the Cross method[24] and the writer's simplified method[25] has the designer had available a simple, rapid, and understandable method of secondary stress analysis. Based upon the assumption that the "best standards of practice" will continue to require the consideration of secondary stresses in design, there is reason to believe that "standard practice" will be developed to include the introduction of secondary stresses into bridge design at increased working stresses. In the past, the question as to whether secondary stresses should be introduced into a design has been largely academic, because of the lack of a convenient method of secondary stress analysis.

The entire argument of the paper is based upon the premise that secondary stresses increase in direct proportion to primary stresses until the yield point of the material is reached at one face of the member, after which an increase in primary stress is not accompanied by an increase in secondary stress. Two reasons are offered to explain this phenomenon. The first (with which the writer agrees) is that local plastic deformation will occur, which will redistribute the stress over the cross-section and maintain (temporarily) the extreme fiber stress at the yield point. The second (with which the writer disagrees), is made clear by the following quotation in reference to Fig. 3:

"Since the angular rotations at the ends of a member can increase no more rapidly than the axial loads, it will be impossible even with high secondary stresses for the extreme fiber stress to exceed the yield-point value without the distortion increasing far beyond any value reasonably to be expected in a truss in service."

The title of this paper, however, indicates that the study relates to ultimate strength, and the deflections, the angular rotations, and the proportional secondary stresses will be several times as large at failure as those presupposed by the authors in the foregoing quotation.

Evidently, this entire matter needs to be clarified by an unqualified statement as to whether the structure is to be designed for working conditions,

[24] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), pp. 1–156.
[25] *Loc. cit.*, Vol. 99 (1934), pp. 610–669.

for an intermediate condition in which stresses will remain at the yield point during a considerable increase of load, or for those conditions existing at failure. If the authors select the intermediate condition unqualifiedly, they will approach more nearly to a justification of their hypothesis that secondary stresses are of little importance in design. However, they will then face the criticism that their structures designed with a standard factor of safety upon the yield point have far less reserve strength above the yield point than the designer has been accustomed to expect. This follows because joint rotations and secondary deformations increase more rapidly than the loads as soon as the direct stresses exceed the elastic limit of the material, and the authors have suggested no provision for secondary stresses in the design.

Considerable emphasis is given to the fact that stresses above the yield point are not unusual elsewhere in structural design. Of course they are not, although there is still a considerable group of engineers who object to plasticity as a basis for structural design. Nevertheless, when the authors declare that stress concentrations above the elastic limit probably exist around rivet holes, they should not forget that their suggestion of the entire neglect of secondaries assumes that this severely deformed material around the rivet holes will be able to continue to flow in order to reduce secondary stresses without developing incipient cracking. The reference to increased allowable bearing stresses on pins and rivets must be considered together with the obvious fact that the highly stressed material is restrained against flow by low stressed adjoining material. Professor Swain made this point clear by stating[26] that a cannon ball cannot possibly fail under hydraulic pressure.

An important factor not considered by the authors would seem to be the possibility of reversal upon accumulative plastic deformation. If reversal can be shown to be sufficiently important to over-stress, alternately, the opposite sides of a member, such accumulative distortion might be serious. Another obviously important consideration is the instantaneous stresses produced by live load impact and vibration. Since there would be no possibility of reducing such stresses by plastic flow, they could not be placed in the same category as the secondary stresses caused by dead load. Although it is difficult to visualize impact as producing deflection and joint rotation, there is reason to believe that the shock effect would increase both secondaries and primaries. In fact, since the secondary stresses do not support the load, it is easy to visualize the possibility that they might be doubled by the individual vibration of the member which could act as an instantaneous application.

In summarizing, the writer would like to express the opinion that this paper will be of great value in opening the entire field of secondary stresses and proper working stresses for discussion. However, he cannot at present agree that the paper justifies, entirely, its hypothesis that secondary stresses should be completely neglected. The probability of high secondaries near the ultimate load, the danger of over-dependence upon plastic deformation

[26] "Structural Engineering—Strength of Materials", by George F. Swain, Past-President and Hon. M. Am. Soc. C. E., McGraw-Hill Book Co., N. Y., 1924, p. 551.

where other stress concentrations exist, the possibility of cumulative plastic deformation, and the impossibility of any reduction in secondary stresses caused by rapid advance of live load, or by impact and vibration, point to the danger that might be involved in a complete neglect of secondary stresses. Probably all could agree, however, that, if specifications are to be rewritten so as to make it obligatory to introduce secondary stresses into the design, a slightly increased basic working stress, further increased one-third as an allowance for secondary effects, is permissible.

This paper marks the authors as pioneers in pointing to the great possibilities that lie in the future for exploiting the advantages of steel as a plastic material.

L. T. EVANS,[27] JUN. AM. SOC. C. E. (by letter).[27a]—While the subject of secondary stresses has been popular with many writers, all papers prior to this one have left the impression that the analogous eccentric-load relation existed for all increments of load.

For some time the writer has felt that some relief must accrue to highly stressed parts of structural members when a large part of that stress was due to secondary moments. Actual structures substantiate this belief. Several years ago the writer was employed on the design of a large, steel, rigid-frame structure. The girders were trusses with web members that had secondary stresses as great as four times the primary stresses. Since these members were short and "stocky," with an indeterminate length (due to large gusset-plates), it was decided to design the section for the full primary, and one-half the calculated secondary, stress with the elastic limit as the working stress. After erection, the structure was encased in concrete and has been in daily service for several years with no signs of distress.

This paper will be appreciated by engineers who have given secondary stresses considerable thought, but it is hoped that the careless reader will not get the impression that such stresses will "take care of of themselves" and can be ignored.

F. E. FAHY,[28] ASSOC. M. AM. SOC. C. E. (by letter).[28a]—The analysis presented in this paper defines clearly the effect of the relieving local deformations which have been said at times[29] to account, at least in part, for discrepancies (generally on the side of safety) between observed and computed stresses in riveted trusses. The writer has long felt that the calculation of secondary stresses by any of the usual methods involves too many rather arbitrary assumptions to be quantitatively accurate, and that the actual effect of secondary stresses on the members of a heavy riveted truss is closely akin,

[27] Structural Engr., Long Beach, Calif.

[27a] Received by the Secretary January 12, 1935.

[28] Asst. Prof., Civ. Eng., Antioch Coll., Yellow Springs, Ohio.

[28a] Received by the Secretary February 18, 1935.

[29] See, for example, "Stress Measurements on the Hell Gate Arch Bridge," by D. B. Steinman, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXII (1918), p. 1071; and "Investigation of Secondary Stresses in the Kenova Bridge," by G. A. Maney and J. I. Parcel, Members, Am. Soc. C. E., University of Minnesota, *Studies in Engineering*, No. 4, p. 8.

as the authors suggest, to the many cases of theoretically high localized stresses which are of little practical significance.

A rather interesting analogy might be drawn between the action of a riveted truss joint at which secondary moments cause deformations beyond the yield point, and that of a multiple-pin joint. Where a multiple-pin joint does not materially affect the static stability of the framework (at the intermediate support panel points of a continuous or cantilever truss, for instance) it can be used advantageously; it permits the connected members to rotate about their respective pins without incurring any flexural deformations except those due to pin friction. In such a case the stresses remain essentially axial. After the extreme fiber stresses in the connected members pass the yield point of the material, secondary stress action under rigid joint conditions will tend toward the same effect, except, as the authors indicate, in the case of slender compression members bent in single curvature. The points of yielding might be considered as approaching pin-end conditions, about which the members involved "rock" to a final state of static equilibrium.

The foregoing analogy will not apply beyond the first application of load because of strain hardening, but it is suggested as a convenient method of visualizing the action of a truss in which flexural overstress occurs due to secondary bending. (As a matter of convenience the writer will use the term, "overstress," to mean stress beyond the elastic range.) In many cases, incidentally, it is the first loading stage that is final and most important; heavy building trusses, perhaps the worst offenders in the matter of high secondary stresses, are essentially dead load structures, and after the first application of full load has imparted to them a certain deformed condition, will hold that condition under the subsequent steady loads to which they are subjected. The truss outlined in Fig. 23 (*a*) can assume the position shown in Fig. 23 (*b*), in which deformations have been idealized and exaggerated

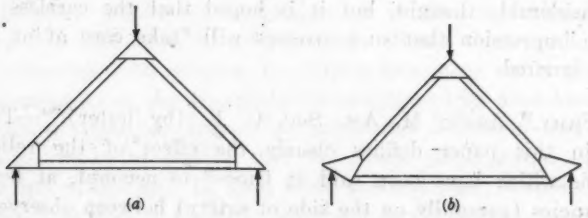

FIG. 23.

for the sake of clearness; and, under the small deformations which actually occur, it can hold that position without harm. It may be noted, also, that the inherent instability of a multiple-pin joint does not appear in an overstressed rigid joint because of the fact that the overstressed members can always offer flexural resistances (until, of course, failure is imminent) even after the extreme fiber stresses have passed the yield point.

It might be suggested that in small bridges, especially railway bridges, which are essentially "live load structures," fatigue might play a fairly important part in causing failure if the combined effect of dead load, live

load, and impact stresses runs as high as the yield point of the material. Current practice properly tends to neglect fatigue as a serious consideration in discussing the utilizable strength of such structures. In the first place, the number of full-load applications which the ordinary bridge receives during its normal life span is so small as to preclude the necessity for considering fatigue either in design or review. Assume, for the sake of argument, however, that this is not the case. All the evidence with which the writer is familiar at the present time seems to indicate that for most ferrous metals the endurance limit for a stress ranging from zero to a maximum value of one sign only is well above the static yield point of the material. The Goodman value is 1.5 times the endurance limit for completely reversed stress, which endurance limit, for the ferrous metals, seems to approach roughly a value equal to one-half the ultimate strength.[30] (In general, for the types of steels commonly used in bridges the endurance limits for completely reversed stress appear to be somewhat less than the yield points, but not enough less than the usual specified minimum yield points[31] to invalidate the comments which follow.) For the ordinary carbon steel commonly used in short-span and medium-span bridges the yield point is also, roughly, equal to one-half the ultimate strength. As the authors have indicated, stresses in bridge members, even with low length-depth ratios, will scarcely exceed the yield point, which means that such stresses can never reach the endurance limit for a range ratio of zero. Furthermore, bridge members which are short enough to be under high secondary moments will almost always be under some dead load stress, in which case the stress range decreases and the endurance limit increases. An exception might be the case of comparatively heavy web members in the center panel or panels of a shallow deck truss. In these members, stress reversals due to secondary bending are quite likely to occur, and reversed primary stresses actually do occur. It would be a rare case indeed in which the resultant primary and secondary stresses in such members in a well-designed structure, would approach anything like the yield point or the endurance limit of the material. The writer simply introduces the subject of fatigue, and the foregoing discussion, as further substantiating the authors' conclusions that, in well-designed structures, overstressing due to secondary bending generally means either a "limited, localized failure not endangering the structure as a whole, or merely undesirable local distortion and permanent set."

The writer does not mean to imply that consideration of secondary stresses should be omitted entirely from the design process. No conditions under which stresses reach the yield point of the material can be considered as other than most undesirable and to be avoided if at all practicable. As far as the members themselves are concerned, however, the writer concurs with the authors in their conclusion that the presence of secondary bending stresses even as high as the yield point are not necessarily damaging; nor are they to be considered as rendering the structure unsafe or poorly designed.

[30] "The Fatigue of Metals," by Moore and Kommers, pp. 160 to 165.

[31] *Loc. cit.*, pp. 128 to 146; also, A..S. T. M. Standards.

A brief discussion of the effects of secondary bending on the joints of a structure seems pertinent, especially in considering secondary stresses due to live load. The presence of secondary moments at a truss joint means that the connection between member and gusset-plate ceases to be a direct tension or compression joint, but must resist flexure as well. Since the intensity of this flexure varies considerably, in the average structure, between the dead load and full live load plus impact conditions, there may be considerable danger of loosening the rivets and generally overstressing the entire group, in a riveted joint, or of causing progressive fracture of a weld.

Consider a tension member made up of two 15-in. 55-lb channels laced or battened together, each channel being riveted to the gusset-plate by means of thirty $\frac{7}{8}$-in. rivets, as in Fig. 24. Assume that the full dead load, live load, and impact stress in the member is 16 000 lb per sq in. of net section, and that the minimum live load stress is zero. (The writer has made this assumption partly for the sake of simplicity and partly because of the fact that under most current specifications members and connections which are subject to reversals of primary stress are quite conservatively designed. Furthermore, such members are usually relatively slender and not likely to be subjected to high secondary stresses. Reversals of secondary bending may heighten the effects noted herein, but the writer has made no attempt to make the examples which follow especially severe, except in so far as secondary bending causes yield-point stresses.)

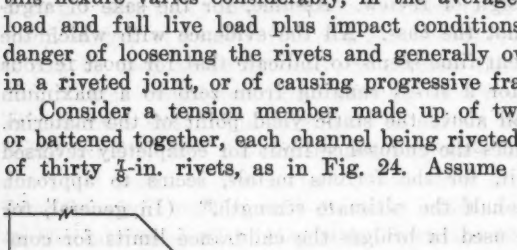

FIG. 24.

Basing the allowance for rivet holes on the 1923 specifications of the American Railway Engineering Association, this gives a total primary stress due to dead load, live load, and impact of 431 000 lb. The total primary stress per rivet is then $\frac{431\,000}{60} = 7\,180$ lb. The gross area of the member is 32.22 sq in., giving a primary stress of 13 400 lb per sq in. on the gross section. The section modulus of the member, based on gross section, is 114.4 in.[3] Assuming a yield point of 32 000 lb per sq in., the secondary bending moment required to produce yielding in the extreme fiber will be $(32\,000 - 13\,400) \times 114.4 = 2\,130\,000$ in.-lb. The moment of inertia of the rivet group (Σd^2) is 3 160 in.,[4] and the distance from the centroid of the group to the extreme rivet is 10.6 in. Then the total stress in the extreme rivet due to the secondary moment which will produce yielding in the member is $\frac{2\,130\,000 \times 10.6}{3\,160} = 7\,130$ lb. Combining this with the

primary stress of 7 180 lb and neglecting the probably small transverse shear which is caused by reversed bending or unequal end moments, gives a resultant total stress of 10 900 lb on the extreme rivet, or 18 100 lb per sq in. in single shear. This is approximately the shearing yield strength of the rivet. If 25% of the total stress is due to dead load, the stress in two of the extreme rivets of the group will vary from 4 500 lb per sq in. to 18 100 lb per sq in. If the rivet has slipped at the first full-load application (as it probably has), it would appear that a relatively small number of live load applications would suffice to loosen it beyond any useful resistance, and thus produce progressive failure of the joint—especially when the highly non-uniform distribution of primary stress among the rivets comprising the group is considered.

Here again, however, the nature of secondary stress action must be remembered. It is essentially due, as the authors state, to angular displacements of the ends of the member. These angular displacements are small, and when the rivets themselves slip, the joint is no longer rigid. Slight rotation of the member is permitted, which has the effect of reducing secondary bending and the rivet stresses caused by it. The extreme rivets will slip at low stresses, once the effect of friction is broken down, producing at once a more uniform distribution of primary stresses throughout the group and a reduction in secondary bending. If the connected member has been deformed to the yield point, but not through or beyond it (say, to a point slightly to the right of Point *B*, Fig. 2(*a*)), the effect on the rivets may not be damaging, although the sharp rotation occurring at the overstressed section will introduce some eccentricity of primary stress. If it has not been so stressed, slip will not reduce the loosening effect of rotation and counter-rotation of the joint. Field and laboratory investigation of the effect of variable bending on groups of rivets in shear would be of value in connection with the subject of secondary stress effects.

Rivet slip may play an appreciable part in reducing secondary bending, which accounts to some extent for discrepancies between observed and calculated stresses. The relieving effect of rivet slip is not to be found in welded joints, although it has a counterpart in the plastic flow which occurs at the ends of such joints even under low loads. Due to the fact that welds are usually located along the edges of the member, the moment of resistance of a welded joint will be relatively higher than in a web-riveted joint, on the usual design assumption of uniform stress distribution through the length of the joint. Even so, however, the stresses in a welded joint at which the connected members are stressed to the yield point by secondary bending, will be high.

The damaging effect of high secondary stresses will likely be most severe on the joints at the ends of non-continuous members, such as the web members, end posts, and end-panel chords of the ordinary types of bridge and building trusses, and on splices in chord members, where such splices are near the panel points. This will be especially true for members in which the ratio of dead load stress to total dead load plus live load plus impact

stress is low or negative. Severe joint stresses are not likely to occur at panel points at which the chord is continuous, except in the web-member connections. In Fig. 25(*a*), for example, $M_1 = M_2 + M_3 + M_4$, and if the

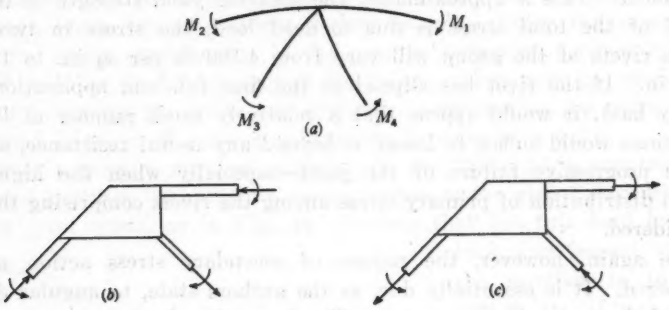

FIG. 25.

chord is continuous only $M_3 + M_4$ must be resisted by the rivets between the chord and the gusset-plates. These rivets are usually considerably in excess of those required for primary stress transmission, and, due to the length of the joint, the group has a high moment of resistance against torsional loads.

Another point which may be worth mentioning is the possible effect of high joint moments in causing severe stresses in the gusset-plates. If the stresses in a gusset-plate due to joint moments are opposite in sign to those due to primary stresses in the members the net effect will be a relief and lowering of the gusset-plate stresses. Such would be the case for the joint of Fig. 25(*b*), under the stresses and moments there shown. If, however, the primary stresses act as shown in Fig. 25(*c*), the secondary stresses add directly to the primary stresses in the gusset-plate. If the result is a tensile stress at a rolled edge of the plate equal to the yield point of the material no serious damage will result, either statically or from fatigue. At a sheared edge there may be some danger of progression of an incipient crack. If the resulting stress at the edge of the plate is compressive and equal to the yield point of the material, buckling may occur unless the unsupported edge of the plate is very short or unless it is stiffened by angles or a diaphragm connecting the usual pair of plates to each other. In the case of fairly large plates under compressive stress along a long unsupported edge, diaphragms or stiffener angles provide an easy and effective means of insuring, to a certain extent, against failure of the plate by buckling. In any design where high secondary stresses are likely to occur gusset-plates stresses should receive some consideration.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

THE HYDRAULIC JUMP IN TERMS OF DYNAMIC SIMILARITY

Discussion

BY HUNTER ROUSE, ESQ.

HUNTER ROUSE,[12] ESQ. (by letter).[12a]—In full accord with modern trends in the science of fluid mechanics, the authors have recognized in this familiar hydraulic phenomenon a physical problem which may be fully understood only through physically correct methods of approach. By means of dimensionless analysis of the phenomenon according to the Froude criterion of dynamic similarity, the hydraulic jump has finally been given definite form and character in terms of a universal parameter, with the determination of general relationships involving jumps of every size and nature.

Too much credit cannot be given for the devotion of many hours to tedious measurement, computation, and discussion, during which the writer was often a privileged witness. Only those who have experienced the high degree of turbulence and accompanying fluctuation in the region of the jump can appreciate the care such measurements entail; yet far from the expected scattering of points, the authors' results show remarkable coordination. Aside from presenting new data covering systematically a great range of conditions, and a correct physical interpretation of these data, the authors' paper serves the excellent purpose of clarifying a number of basic points often confused in previous investigations. The writer adds the following discussion in the hope that it may throw further light upon so important a type of fluid motion.

During the century following Bidone's first description of the jump, hydraulicians were slow to conclude that the popular energy principle had to be abandoned for the principle of momentum before the height of the jump could be computed successfully. Although the application of the momentum principle had been suggested shortly before the middle of the Nineteenth Century by Bélanger, emphasized later by Bresse, and finally discussed at length by Unwin in the Encyclopedia Britannica in 1880, as late as 1916

NOTE.—The paper by Boris A. Bakhmeteff, M. Am. Soc. C. E., and Arthur E. Matzke, Jun. Am. Soc. C. E., was published in February, 1935, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

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[12a] Received by the Secretary February 27, 1935.

many publications, both in the United States and abroad, still held persistently to the assumption of constant energy according to the Bernoulli theorem.

Various writers, however, developed empirical equations for the relation of the two depths, believing that energy loss must exist; typical of these is that of Merriman, included in the early editions of his text on hydraulics.[13] Whatever the relationship used, abundant experimental results were quoted by the different writers, the best-known data being those of Bidone (1818), Darcy-Bazin (1856), Ferriday-Merriman (1894), Gibson (1913), and Riegel-Beebe, of the Miami Conservancy District (1917). With the perfection of the indoor hydraulic laboratory in Germany considerable study was given the jump, of which the experiments carried out under Koch at Darmstadt are good examples.[14] Rehbock made constant use of the basic momentum principle in the design of spillway aprons,[15] and the much-quoted thesis of Böss on open-channel flow[16] discussed the position of the jump with regard to channel slope and discharge. It is noteworthy, however, that little mention was made by any of these investigators of the longitudinal profile; nor had successful attempts been made to treat the problem as one of dynamic similarity.

Apparently, Groat and Kennison were the first to make use in this case of a term similar in form to the Froude number,[17] but it remained to Safranez,[18] in a series of experiments conducted in the laboratory of the Technische Hochschule, at Berlin, to undertake a two-dimensional physical study of the problem, in which constant use was made of the foregoing dimensionless term—without realization that it was actually the Froude number. Safranez gave an empirical relationship for the length of the jump, stating, in addition, that it was approximately 4.5 times the down-stream depth and that the slope of the jump was practically constant. (Refer to Figs. 8 and 9 of the authors' paper; these two diagrams deserve careful study,[18a] as they show in comprehensive form the actual variation of the length and the shape of the jump with the Froude number, and the transition to the undular form as this number approaches a minimum.)

Following closely upon the publication of the foregoing papers, a controversy arose abroad over the part played by the surface roller in the apparent energy loss accompanying the jump. The roller was held by several (in particular by Kozeny[19]) to be of negligible influence, and various experimental and theoretical methods are still being used and misused to prove both sides of the case. Einwachter, whose recent paper on the length of the

[13] "A Treatise on Hydraulics," by Mansfield Merriman, M. Am. Soc. C. E., Fourth Edition, John Wiley & Sons, New York, 1894.

[14] "Bewegung des Wassers," von Koch-Carstenjen, Julius Springer, Berlin, 1926.

[15] "Die Verhütung schädlicher Kolke bei Sturzbetten," von Th. Rehbock, *Der Bauingenieur*, 1928, Heft 4, 5.

[16] "Berechnung der Wasserspiegellage," von P. Böss, V. D. I. Verlag, Berlin, 1927.

[17] "The Hydraulic Jump in Open Channel Flow at High Velocity," by Karl R. Kennison, M. Am. Soc. C. E., *Transactions* Am. Soc. C. E., Vol. LXXX (1916), p. 338.

[18] "Wechselsprung und die Energievernichtung des Wassers," von K. Safranez, *Der Bauingenieur*, 1927, Heft 49; "Untersuchungen über den Wechselsprung," *Der Bauingenieur*, 1929, Heft 37, 38; and "Energieverzehrung der Deckwalze," *Der Bauingenieur*, 1930, Heft 20.

[18a] Note correction in paper as published in February, 1935, *Proceedings:* In Fig. 8, the horizontal scale should vary from 0 to 9.0 to correspond with the correct values in Fig. 9.

[19] "Wassersprung und Energieumwandlung," von J. Kozeny, *Wasserkraft und Wasserwirtschaft*, Heft 27, 1932.

jump[20] has been supplemented by as yet unpublished experiments, has attempted to show the importance of the roller by replacing it with a fixed upper boundary and noting the resulting change in depth; Schoklitsch,[21] on the other hand, has analyzed the action of the roller from the viewpoint of regions of discontinuity (sources of turbulence) between the roller and the actual flow resulting in turbulent interchange of momentum.

The most recent departure from traditional research has been the endeavor of Engel[22] to study the jump as a three-dimensional phenomenon in the diverging section of a Venturi flume. It was by Engel that the Boussinesq number was devised, thus incorporating in a universal parameter characteristic of gravitational action a width term having no bearing whatever upon the force of gravity.

The authors have distinguished carefully between the significance of the Froude number and of the Reynolds number, stating that the Froude number is the correct criterion in cases of gravitational action only, and the Reynolds number in the case of viscous action only, and that it is practically impossible to obtain complete dynamic similarity if both viscous and gravitational forces are present to an appreciable degree.

It is quite possible that, in the past, the loss of energy accompanying the jump has been the indirect cause of the failure to consider the jump in terms of the Froude number, for the hydraulic engineer has become prone to think of energy loss as synonymous with conversion into heat, the latter being possible in fluids only through viscous action. Thus, it would seem at first thought that viscous forces are by no means negligible in the hydraulic jump. In fact, discrepancies between computed and measured values of depths are generally attributed to friction losses along the bottom and sides of the channel. Actually, however, such losses are probably extremely small over the short length of flow involved; the discrepancy arises rather from a failure to consider the velocity defect along bottom and sides in computing the true mean velocity head; it is a known fact that such failure may cause considerable error in measuring the actual energy of flow, in particular since this defect is exaggerated by such abrupt deceleration of flow as occurs in the jump.

Clarity in the matter of energy loss may be attained only through a revision of the impressions received in elementary courses in hydraulics. "Loss" of hydraulic energy is seldom synonymous with immediate conversion into heat energy; in fact, it is only in the case of pure laminar flow that such immediate conversion takes place. In turbulent flow through smooth pipes, for instance, similar direct loss occurs in the laminar boundary film adjacent to the pipe walls, but within the turbulent region the situation is quite different.

In the familiar Bernoulli theorem, the specific energy of flow is expressed as the sum of velocity head, pressure head, and elevation. What the theorem

[20] "Berechnung der in der Wehrbreite gemessenen Langenausdehnung von Deckwalzen," von J. Einwachter, *Wasserkraft und Wasserwirtschaft*, Heft 14, 21, 1932; Heft 17, 1933.

[21] "Ueber die Energievernichtung durch Walzen," von A. Schoklitsch, *Die Wasserwirtschaft*, Heft 16, 17, 1932.

[22] "Non-Uniform Flow of Water," by F. Engel, *The Engineer*, April 21 and 28, and May 5, 1933; and "Wassermengenmessung mit offenen seitlich eingeschnürten Kanälen," von F. Engel, *Zeitschrift* des Verein Deutscher Ingenieur, December 2, 1933.

fails to include is a fourth term expressing the energy of turbulence, often of great magnitude, which the hydraulician has had as yet no means of measuring or of utilizing; this missing term should denote the kinetic energy of rotation, since the velocity head includes only the kinetic energy of translation. It is the conversion to this turbulent head which corresponds in large measure to the apparent energy "loss" caused by valves, "stilling" racks, abrupt changes in section, flow through rough pipes, and by the hydraulic jump. In each of these cases the hydraulic "loss" or formation of excessive turbulence is largely or entirely independent of viscous action. Although turbulence itself may be reduced in intensity only through viscous shear and accompanying conversion into heat, this latter process is by no means immediate, but on the contrary is extremely gradual. Hence, it is that the resistance coefficient for rough pipes at high Reynolds numbers (where turbulence is caused by the action of the roughness projections) is a constant term, independent of changes in viscosity; for the same reason the energy "loss" or conversion from energy of translation into energy of turbulence in the hydraulic jump is practically independent of viscosity and hence is not a function of the Reynolds number.

Were it possible to express energy of turbulence as a fourth term in the Bernoulli equation, it would follow at once that the jump could be treated correctly by means of the energy principle, since the actual energy of the individual particles is practically unaffected by thermal loss during the short time of passage through the jump. It is to be remarked that the general problem of turbulent energy is of outstanding importance in modern fluid research.

The question of the part played by the surface roller should be quite clear once this conception of turbulence is understood. The roller is as indispensable a part of the direct jump as the turbulent wake caused by flow past a valve is an inseparable part of the function of the valve. Those who have observed carefully the behavoir of the roller through the glass walls of an experimental flume will recall vividly enough the fact that the roller is not the idealized, distinctly outlined region of flow shown schematically in illustrations, but an active participant in the diverging flow. In other words, no portion of fluid involved at any instant in the reverse motion of the roller remains in the roller for more than a brief space of time; instead, violently rotating masses of this roller are torn away constantly into the live stream below, this continuous process resulting in the high degree of turbulence—and hence of apparent loss—which is characteristic of the phenomenon.

It should be obvious that such recent expedients as omitting the roller from the physical analysis of the phenomenon or replacing it by a sloping upper boundary held in place above the flow are quite beside the point. On the other hand, although the roller is inseparably associated with the jump, its length is not a true measure of the jump itself; nevertheless, European experimenters have often located the apparent end of the jump by determining visually that point at which the reversal of surface movement terminates. It is to the credit of the authors that their analysis is particularly definite and correct in all these troublesome points.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

A DIRECT METHOD OF MOMENT DISTRIBUTION

Discussion

BY MESSRS. L. E. GRINTER, W. H. HUANG, FELIX H. SPITZER, HAROLD E. WESSMAN, EGOR P. POPOFF, AND L. T. EVANS

L. E. GRINTER,[5] ASSOC. M. AM. SOC. C. E. (by letter).[5a]—In this paper the Engineering Profession is given a clear and concise statement of another method of analyzing continuous frames that is based upon the Cross method of balancing fixed-end joint moments. This method has been in use by associates of Professor Cross since about 1926, and a brief discussion of it has been published.[6] Several other papers have appeared, describing similar procedures. The writer has published two references to the method. The first will be found in his discussion of the paper[7] by Professor Cross where mention is made of an "approximate method of balancing moments where the carry-over moment is determined from the restraint at the far end of the member." The second occurs in a discussion on methods of checking rigid frame analysis.[8]

The foregoing references are mentioned, not with the idea of establishing prior authorship of the method (which obviously goes back to the original work of Professor Cross), but rather to establish the fact that the writer speaks from a background of experience when suggesting that the method described by Mr. Lin is of limited usefulness. It was introduced into the writer's graduate and undergraduate classes from 1929 to 1932. It has since been withdrawn from the undergraduate classes because experience has shown that its usefulness is limited and because it proved confusing to the student.

A study and comparison of Figs. 9, 10, and 11 of the paper, reveals a rapid increase in the volume of recorded calculations on successive illustra-

NOTE.—The paper by T. Y. Lin, Jun. Am. Soc. C. E., was published in December, 1934, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[5] Prof., Structural Eng., Agri. and Mech. Coll. of Texas, College Station, Tex.

[5a] Received by the Secretary January 10, 1935.

[6] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, Members, Am. Soc. C. E., see Section Heading, "Moment Distribution Using a Constant for End Rotation."

[7] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 13.

[8] *Engineering News-Record*, August 23, 1934, p. 249.

tions. There are sixteen recorded results on Fig. 9, twenty-two on Fig. 10, and fifty on Fig. 11. The explanation is evidently found in the fact that the number of loaded spans has been increased from one to three. There would be a far smaller rate of increase in the number of recorded results by the original Cross method. The total number in the most complex case usually could be reduced to less than fifty, and still yield satisfactory accuracy. This fact is well illustrated by Fig. 14 which is the analysis of a frame similar to that of Fig. 11 by the procedure of balancing moments used by the writer. Incidentally, the five-place results recorded in Fig. 11 seem to be entirely unjustified and, in fact, their use is in conflict with the basic idea of the method which is most useful as an approximate method of moment distribution.

A comparison of the number of tabulated results as discussed in the previous paragraph is unfair to the original Cross method because the preliminary results recorded by Mr. Lin in Fig. 8 require repeated substitutions into formulas, whereas each result recorded by Professor Cross is obtained by a single setting of a slide-rule—or mentally by dividing a number in half.

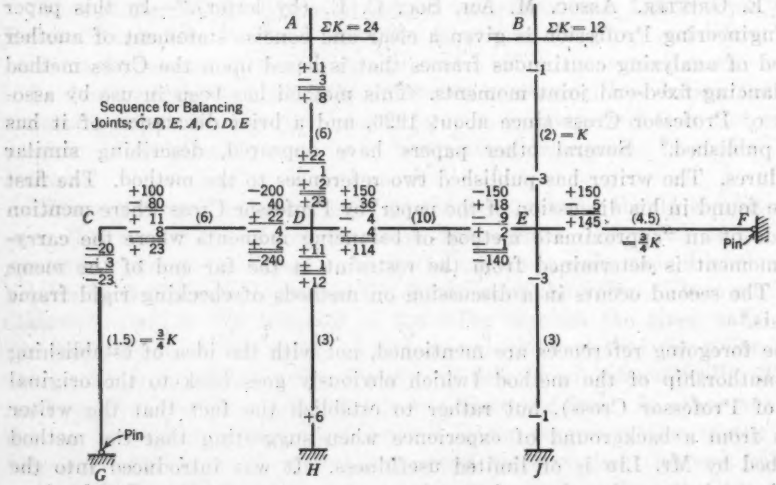

FIG. 14.—ANALYSIS OF A FRAME WITH THREE LOADED SPANS BALANCING FIXED-END MOMENTS.

Hence, in the majority of cases, the writer has found the standard method of balancing moments to be the simpler procedure, and invariably so when loads occur on several spans. The method described by Mr. Lin is advantageous when one is "running out" moments into unloaded spans in the manner illustrated by Fig. 9.

In order to make the paper as useful as possible, Fig. 15 is presented, which the writer developed several years ago for the use of his students. The nomenclature will be clear if it is understood that v is a factor such that Kv represents the relative resistance of a member to end rotation

taking into account its condition of restraint at the far end. Hence, Kv is K_m as defined by Mr. Lin. Obviously, the curves are intended for use in the analysis of frames with prismatic members. The necessary formulas for

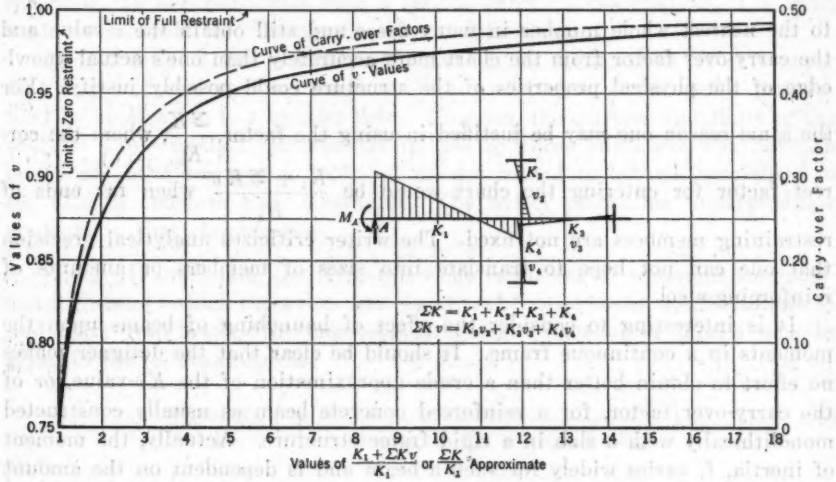

FIG. 15.—CARRY-OVER FACTORS AND v-VALUES FOR MOMENT DISTRIBUTION.

plotting the curves are developed from the data of Fig. 16, as follows:

$$\theta_1 = \frac{M_A L}{4 E I} \text{ and } \theta_2 = \frac{M_A L}{3 E I} - \frac{M_B L}{6 E I}, \text{ from which,}$$

$$v = \frac{\theta_1}{\theta_2} = \frac{\frac{M_A}{4}}{\frac{M_A}{3} - \frac{M_B}{6}} = \frac{1}{\frac{4}{3} - \frac{2}{3}\frac{M_B}{M_A}} \quad \ldots\ldots\ldots\ldots (5)$$

or, the carry-over factor equals,

$$\frac{M_B}{M_A} = \frac{4v - 3}{2v} \quad \ldots\ldots\ldots\ldots (6)$$

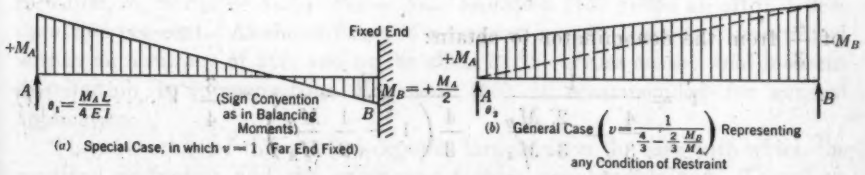

FIG. 16.—SIGNIFICANCE OF v-VALUES.

In using the curves of Fig. 15 one need not attempt to obtain the v-value or the carry-over factor with great accuracy. The primary advantage of any method of moment distribution is the ease and speed with which one can obtain results to a reasonable standard of accuracy. It is evident that the

physical characteristics of the structure make it impossible for one to determine the actual bending moment in a concrete frame to a greater degree of accuracy than ± 5 per cent. Hence, one can approximate the factor, $\frac{\Sigma K}{K_1}$, to the nearest whole number in many cases and still obtain the v-value and the carry-over factor from the chart more accurately than one's actual knowledge of the physical properties of the structure could possibly justify. For the same reason one may be justified in using the factor, $\frac{\Sigma K}{K_1}$, where the correct factor for entering the chart would be $\frac{K_1 + \Sigma Kv}{K_1}$ when far ends of restraining members are not fixed. The writer criticizes analytical precision that one can not hope to translate into sizes of members or amounts of reinforcing steel.

It is interesting to consider the effect of haunching of beams upon the moments in a continuous frame. It should be clear that the designer makes no effort to obtain better than a crude approximation of the Kv-value, or of the carry-over factor, for a reinforced concrete beam as usually constructed monolithically with a slab in a rigid frame structure. Actually, the moment of inertia, I, varies widely for such a beam and is dependent on the amount and sign of the bending moment because of the change from a rectangular section to a T-section at the point of contraflexure. It seems almost absurd to involve one's calculations to care for the effect of a small haunch—which may change a carry-over factor from 0.5 to 0.47—when an equal irregularity has been passed over without consideration. A study of the discussions of the writer's paper[9] on "Wind Stress Analysis Simplified," will make this observation clear.

Mr. Lin's idea of direct moment distribution can be used as a check upon the original Cross method of balancing moments. Since moments must be distributed according to the Kv-values, it is possible to check the total values of the distributed moments (balancing moments plus carry-over moments, but not including original fixed-end moment) by comparing their relative values to the relative Kv-values at the joint. For this purpose, it is convenient to revise the expression for v by removing a constant factor of $\frac{4}{3}$ from the denominator to obtain:

$$v = \frac{1}{\frac{4}{3} - \frac{2}{3}\frac{M_B}{M_A}} = \frac{1}{\frac{4}{3}\left(1 - \frac{1}{2}\frac{M_B}{M_A}\right)} = \frac{3}{4}\ V \quad \ldots\ldots\ldots (7)$$

in which, $V = \frac{1}{1 - \frac{1}{2}\frac{M_B}{M_A}}$. Hence, the distributed joint moments must also be in proportion to the KV-values. This is a complete check upon the balancing procedure, but is not a check upon the original fixed-end moments.

[9] *Transactions*, Am. Soc. C. E., Vol. 98 (1934), pp. 610–669, Fig. 9 and Table 11.

Neither does it check the moments where there is a joint translation in the structure. The writer prefers to check his results by statics and then to draw the deflected structure to see that the requirements of continuity are satisfied. In one particular case a serious error in the analysis of a wind bent was located[10] by plotting the column deflections.

This discussion has pointed out some of the applications of this paper that may be found useful. The individual preferences of the reader may extend its application to a broader field. However, the extreme simplicity of the original Cross method recommends its use by those engineers who have an occasional need for such a tool, but do not intend to become specialists in the field of indeterminate structures. The writer places his undergraduate students in that classification.

W. H. HUANG,[11] ESQ. (by letter)[11a].—The fundamental equations for modified stiffnesses (which the writer has elsewhere[12] termed "conjugate" stiffnesses) and carry-over factors for members of constant moment of inertia may be given in the following forms:

$$R_{ab} = S_{ab}\left(1 - \frac{S_{ab}}{4\,(S_{ba} + R_{bc} + R_{bd} + R_{be})}\right) \ldots\ldots\ldots (8)$$

and,

$$\gamma = \frac{1}{2 + 1.5\,\dfrac{S_{ab}}{R_b - R_{ba}}} \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

in which R_{ab} = the modified stiffness of Member AB at End A; $S_{ab} = S_{ba}$ = the stiffness of Member AB or BA; R_b = the summation of the modified stiffnesses of Members BA, BC, BD, and BE, at End B; and γ = the carry-over factor of Member AB at End A.

Instead of using the cumbersome Equation (8), or the author's Equation (3), which in certain cases can only be solved by successive approximations, the writer suggests[12] the use of the following approximate equation;

$$R_{ab} = S_{ab}\left(1 - \frac{S_{ab}}{4\ S_b}\right) \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in which, $S_b = S_{ba} + S_{bc} + S_{bd} + S_{be}$. Equation (10) yields an error of less than 2.08 per cent. As the stiffness of a member usually cannot be determined within an accuracy of 2%, and as the effect of the errors on the final moment distribution is compensating, Equation (10) is recommended for general application.

The advantage of this method depends largely upon the ease with which the modified stiffnesses and the carry-over factors can be obtained. Therefore, a graphical method for solution is of value, and the writer uses nomographs in connection with Equations (9) and (10).

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[10] *Transactions*, Am. Soc. C. E., Vol. 98 (1934), p. 663.

[11] Scholar of National Tsinghua Univ., Peiping, China; Graduate Student, State Univ. of Iowa, Iowa City, Iowa.

[11a] Received by the Secretary January 5, 1935.

[12] *Journal*, The Chinese Inst. of Engrs., Vol. 9, No. 5.

FELIX H. SPITZER,[13] ASSOC. M. AM. SOC. C. E. (by letter).[13a]—Justly, the author may claim for his presentation the advantage of superior convenience as a tool of structural design in comparison with the original Cross method; but he cannot maintain his claim when his method is compared with the modification of the Cross method introduced by L. E. Grinter, Assoc. M. Am. Soc. C. E.,[14] which has reduced labor to such an extent that frequently the calculation of stiffnesses of members and joints and other preliminary work common to all methods, becomes the major part of the work to be performed.

Due to the application of "modified" coefficients and of additional equations, although very simple, Mr. Lin's modification has lost much of the simplicity so characteristic of the Cross method. This desirable quality of a simple physical process easily visualized and remembered, is even more apparent in the Grinter modification, where the joints, which visual inspection finds to be most unbalanced, are successively released and balanced until an accuracy sufficient for the purpose is reached.

There is scarcely need to sacrifice simplicity to exactness when for all practical purposes the difference is small enough to be of no importance.

HAROLD E. WESSMAN,[15] ASSOC. M. AM. SOC. C. E. (by letter).[15a]—The so-called "direct" method of moment distribution is presented in this paper in a very satisfactory manner. Several engineers have previously indicated their preference for a single, direct distribution and have given general equations which evaluate actual end restraints and lead directly to the final moments.

Of the various forms, however, the writer prefers Mr. Lin's presentation of equations for correct stiffness and carry-over factors. His derivation proceeds quite logically in accordance with the basic sequence of operations so nicely outlined by Professor Cross in his fundamental method, a treatment which emphasized concepts in terms of simple words and numbers rather than of general algebraic equations.

A designer, analyzing a structural framework with a variety of members subject to a number of loading conditions, will find it worth while to become familiar with this direct method, or some other abbreviated procedure[16]; but in classroom practice, the writer feels that better results are obtained when moment distribution is taught with a minimum number of equations. That, of course, means teaching the basic method. The two equations proposed by the author, although simple, are not easily remembered unless one is using this tool of analysis constantly. Incidentally, Mr. Lin should have

[13] Structural Engr., Dept. of Public Works, San Francisco, Calif.

[13a] Received by the Secretary January 12, 1935.

[14] Discussion of the paper by Hardy Cross, M. Am. Soc. C. E., entitled, "Analyses of Continuous Frames by Distributing Fixed-End Moments", *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 11.

[15] Associate Prof. of Structural Eng. and Mechanics, State Univ. of Iowa, Iowa City, Iowa.

[15a] Received by the Secretary January 17, 1935.

[16] See methods proposed in "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, Members, Am. Soc. C. E.

stated that it is necessary to remember three equations, rather than two. One must also keep Equation (1) for R_{ba} in mind.

In the "Synopsis" of his paper, the author might have qualified the term, "exact." The method is exact for any structure of the type in Fig. 17, which does not contain closed structural panels and in which the conditions of restraint of all external terminals, such as those at *A*, *B*, *D*, *E*, *G*, and *H*, are completely known. The method is not exact for any structure of the type in Fig. 18, which contains a closed structural panel, *CDGF*. One must

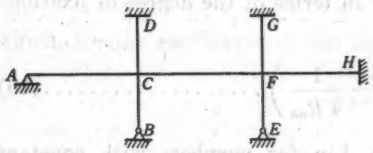

FIG. 17.

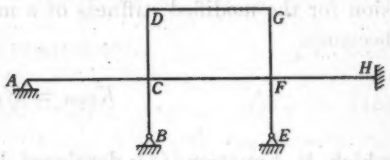

FIG. 18.

assume an initial value for R_{dc} or R_{gf}. The author notes a similar case in his Example 4. It is further illustrated by his assumption of values for R_{ad} and R_{be} in Example 1.

Ordinarily, one can estimate these values closely enough so that the error involved in final moments is of no practical significance. It is also well to keep in mind that the error involved in moment distribution by terminating computations at the end of two cycles of distribution is ordinarily of no consequence in design.

EGOR P. POPOFF,[18] JUN. AM. SOC. C. E. (by letter).[18a]—The modification, presented by Mr. Lin, of the well-known Cross method of analysis contains material of considerable interest. It is timely to note that quite similar modifications have been proposed by a number of writers.

Mr. Lin uses the method of moment distribution, entirely, for the derivation of his equations for the modified stiffnesses of the members and for the "carry-over" factors. Other writers use the slope deflection method; but fundamentally the two methods are closely related.

Analyzing the work of three writers in this field,[19] it may be noted that a mere algebraic transformation is sufficient to put all the results into an identical form, if a common definition can be written for the "factor of restraint," R_{ba}. The expression used by Mr. Lin in Equation (1) is the restraining effect of the several members at a joint on the member under consideration. The factor of restraint is variously defined: Mr. Lin defines it by Equation (1), whereas others define it by $\sum \frac{K_{bnm}}{K_{ab}}$, and $\frac{\Sigma K_{bnm}}{K_{ab} + \Sigma K_{bnm}}$ (expressed in Mr. Lin's notation). Mr. E. B. Russell,[19] defines it as,

$$R'_{ba} = \sum \frac{K_{bnm}}{K_{ab}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

[18] Asst., Dept. of Civ. Eng., California Inst. of Technology, Pasadena, Calif.
[18a] Received by the Secretary January 16, 1935.

whereas, L. T. Evans,[19] Jun. Am. Soc. C. E., and T. F. Hickerson,[19] M. Am. Soc. C. E., define it as,

$$f_{ba} = \frac{\Sigma K_{bnm}}{K_{ab} + \Sigma K_{bnm}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Of the foregoing definitions, perhaps Equation (9) is the most logical, because it has a value of 1 for the fixed conditions and 0 for the hinged condition.

Considering, for simplicity, members of uniform cross-section, the expression for the modified stiffness of a member in terms of the degree of fixation[20] becomes,

$$K_{bnm} = K_{bn}\left(1 - \frac{1}{4\,R_{nb}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

which is Equation (3) developed by Mr. Lin for members with constant moment of inertia. The extension to a more general case is obvious.

The idea of the modified stiffness of a member reveals a new concept. Thus, in addition to the dependence on the length and cross-section, the true stiffness is also dependent upon the degree of restraint at the far end of the beam.

For the special case of a member with a uniform cross-section, the following has been offered,[21]

$$\gamma_{abm} = \frac{1}{2}\,\frac{R_{ba} - 1}{R_{ba} - \frac{1}{4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

which is Equation (4) developed by Mr. Lin.

In the modifications cited for comparison herein, the equations developed are for the general cases, and many graphs and tables are prepared; but still other writers define some similar coefficients, such as R, differently, and thus the transformation into the form of equations as given by Mr. Lin is not quite as obvious as for the case shown, which is by far the most important one.

The method seems to be somewhat longer for the simpler problems than the usual Cross method. It appears to be especially inconvenient for closed frames as in Example 4 where several trials may be necessary to check the value of R originally assumed. However, this group of modifications is of practical significance for the cases in which there are several types of loading for the same structure. The method is also satisfactory for the simpler cases with the assistance of the extensive tables now available.[19]

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[19] "Handbook of Rigid Frame Analysis," by L. T. Evans, Jun. Am. Soc. C. E., Edwards Brothers, 1934; "Structural Frameworks," by T. F. Hickerson, M. Am. Soc. C. E., Univ. of North Carolina Press, 1934; and "Analysis of Continuous Frames," by E. B. Russell, Ellison and Russell, 1934.

[20] "Handbook of Rigid Frame Analysis," by L. T. Evans, Jun. Am. Soc. C. E., Edwards Brothers, 1934 (In Equation (174), let $f = 1 - \frac{1}{R}$).

[21] "Structural Frameworks," by T. F. Hickerson, M. Am. Soc. C. E., Univ. of North Carolina Press, 1934, Equation (13), where, again, let $f = 1 - \frac{1}{R}$.

L. T. EVANS,[22] JUN. AM. SOC. C. E. (by letter).[22a]—An interesting addition to engineering literature is contained in this paper, but it is to be regretted that Equations (1), (2), and (3) are incorrect. The first error is introduced in the assumption that a moment equal to $K_{ab} = \frac{I}{L}$ applied at End A will produce a unit angular deflection at A. Applying the modified slope-deflection[23] equations to End A:

$$M_{AB} = E\,K_c\,(C_1\,\theta_A + C_2\,\theta_B - [C_1 + C_2]\,R) - C_{AB} \quad \ldots\ldots(12)$$

which, for the conditions of the problem, reduces to:

$$M_{AB} = C_1\,E\,K_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

in which, K_c represents $\frac{I_c}{L}$ for a specified point of the beam, and C_1 is the beam coefficient, the value of which is determined by the variation of moment of inertia of the member. For a constant moment of inertia, $C_1 = 4$.

Since an external moment applied to a joint will be resisted by the moments in the members which will be in direct ratio to the "modified stiffness" of each member, Equation (1) should be:

$$R_{ba} = \frac{C_1\,K_{ba} + \Sigma\,K_{bnm}}{C_1\,K_{ba}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

The author has been inconsistent in using modified stiffness for Members $B1$, $B2$, and $B3$, but not for Member BA. In order to obtain the correct results for the problem as stated, the modified stiffness of Member BA must also be used. Under the assumed condition of full restraint[24] at End A in Step 2 the modified stiffness would be: $K_{ban} = C_1\,K_{ab}$ and Equation (2) becomes:

$$M = -\frac{C_1\,K_{ab}\,\gamma_{ab}}{R_{ba}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

which changes Equation (3) to:

$$K_{abn} = C_1\,K_{ab}\left(1 - \frac{\gamma_{ab}\,\gamma_{ba}}{R_{ba}}\right) \quad \ldots\ldots\ldots\ldots\ldots(16)$$

Expressed in terms of the degree of restraint at End B of Member AB (that is, f_{ba}) Equation (16) should read:

$$K_{abn} = K_{AB}\left[C_1 - (1 - f_{ba})\,\frac{C_2^2}{C_3}\right] \quad \ldots\ldots\ldots\ldots\ldots(17)$$

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[22] Structural Engr., Long Beach, Calif.

[22a] Received by the Secretary January 22, 1935.

[23] See "Modified Slope-Deflection Equations", by L. T. Evans *Journal*, Am. Concrete Inst., October, 1931, pp. 109–130.

[24] For a complete development and application of the method of restraints, see "Handbook of Rigid Frame Analysis", by L. T. Evans.

which reduces to:

$$K_{abn} = K_{AB}\,(3\tfrac{1}{2} + f_{ba}) \qquad \ldots\ldots(18)$$

for constant moment of inertia, and Equation (4) for constant moment of inertia is:

$$\gamma_{abm} = \frac{2\,f_{ba}}{3 + f_{ba}} \qquad \ldots\ldots(19)$$

It is to be noted also that the author's formulas, Equations (1), (2), and (3), apply only when the far end of Member AB is fixed, which introduces another error when applied to continuous structures. Equations (17) to (19) are general and apply for any degree of restraint at the far ends of the members.

The value, $R_{cd} = 1.25$, in the frame of Fig. 8 is equivalent to:

$$R = \frac{K_{CD} + \dfrac{K_{CG}\,(3 + f_{gc})}{C_1}}{K_{CD}} = \frac{6 + \dfrac{2 \times 3}{4}}{6} = 1.25$$

The correct value is:

$$R = \frac{6\,(3 + f_{dc}) + 2 \times 3}{6\,(3 + f_{dc})}$$

which could yield $R = 1.25$ only for the special case of $f_{dc} = 1$. Members DA, DE, and DH do not fix End D of CD; in other words, f_{dc} is less than unity and, therefore, the correct value of this R is greater than 1.25. The author does not give sufficient data concerning Member DE to allow the computation of f_{dc}.

The correct expression for R_{dc} is:

$$R_{dc} = \frac{C_1\,K_c\,(0.51 + 0.49\,f_{cd}) + 6 \times 3.2 + 3 \times 4 + 6\,(3 + f_{ad})}{C_1\,K_c\,(0.51 + 0.49\,f_{cd})} \qquad \ldots(20)$$

Equation (20) takes into consideration both the beam coefficients and the actual degree of restraint at the far ends of the members.

Although one may begin with the fixed-end moments and make a correction, it seems more direct and logical to use a moment that takes into consideration the actual degree of restraint at both ends of the member. For example, consider the frame shown in Fig. 8. Let it be required to compute the moments, M_{AB} and M_{BA}, due to a concentrated load of 1 000 lb applied at the center of Span AB. The fixed-end moment is $\frac{PL}{8} = 3\,750$ ft-lb. The degree of restraint at End A of AB is $f_{ab} = \dfrac{3 \times 3.8}{3 \times 3.8 + 5 \times 4} = 0.363$, and, in a similar manner, $f_{ba} = 0.566$.

Now, let $D_4 = 3 + f_{ab} + f_{ba} - f_{ab}\,f_{ba} = 3.723$; and,

$$M_{AB} = -\frac{2\,f_{ab}\,(3 - f_{ba})}{D_4} \times C_{AB} = -1\,780 \text{ ft-lb}$$

and,

$$M_{BA} = + \frac{2 f_{ba} (3 - f_{ab})}{D_4} \times C_{BA} = + 3\,005 \text{ ft-lb}$$

These moments have the same values that would be obtained by the distribution methods after all joints have been "unlocked," all distributions completed, and the results totaled. It is seen that the "method of restraints" provides an easy, exact, and rapid solution and is especially adapted to problems of partial restraint, such as indicated at the bases on the columns of Fig. 8.

In the author's "Conclusion" it is stated that "the effect of fixity of supports is shown by the value, R_{ba}, and the effect of haunching is shown by the value, γ_{ab} γ_{ba}." This statement is in error because both the values, R_{ba} and γ_{ab} γ_{ba}, are influenced by both the degree of restraint at the ends of the members and by the moment of inertia variation.

Although the greater part of this discussion has been devoted to pointing out inconsistencies, the writer wishes to make it clear that this has been done to help Mr. Lin build up his work and not to destroy. The writer realizes from his own experience that the author has used considerable time in the development of the material of the paper and deserves credit. Advancement of the profession can occur only through the hard work of the members.